Chapter 9b: Numerical Methods for Calculus and Differential Equations

• Initial-Value Problems
• Euler Method
• Time-Step Independence
• MATLAB ODE Solvers
Initial-Value Problems

Consider a skydiver falling from an airplane. A Free-Body Diagram of the skydiver is shown:

Newton’s First Law is given by:

\[ \sum F = ma \]

\[ mg - F_D = m \frac{dv}{dt} \]

Substitute an expression for the Aerodynamic Drag Force:

\[ mg - \frac{1}{2} \rho v^2 AC_D = m \frac{dv}{dt} \]
Initial-Value Problems

\[ mg - \frac{1}{2} \rho v^2 A C_D = m \frac{dv}{dt} \]

This is a First-Order Ordinary Differential Equation. In particular, it is called an Initial-Value Problem, because it is solved by knowing an Initial Value of the Dependent Variable. For instance, we can assume that the Downward Velocity of the skydiver was initially zero:

\[ v = 0 \text{ at } t = 0 \]
Euler Method

\[ mg - \frac{1}{2} \rho v^2 AC_D = m \frac{dv}{dt}; \quad v = 0 \text{ at } t = 0 \]

This Initial-Value Problem can be solved for the skydiver’s velocity as a function of time by using the Euler Method, which starts with the Definition of the Derivative. The derivative of the velocity is:

\[ \frac{dv}{dt} = \lim_{\Delta t \to 0} \frac{v(t + \Delta t) - v(t)}{\Delta t} \]

The derivative of the velocity can be approximated by allowing \( \Delta t \) be a small (but finite) value:

\[ \frac{dv}{dt} \approx \frac{v(t + \Delta t) - v(t)}{\Delta t} \]
Euler Method

\[
\frac{dv}{dt} = \frac{1}{m} \left( mg - \frac{1}{2} \rho v^2 AC_D \right)
\]

\[
\frac{v(t + \Delta t) - v(t)}{\Delta t} = \frac{1}{m} \left( mg - \frac{1}{2} \rho v^2 AC_D \right)
\]

\[
v(t + \Delta t) = v(t) + \frac{\Delta t}{m} \left( mg - \frac{1}{2} \rho [v(t)]^2 AC_D \right)
\]

Knowing the Initial Condition for the velocity, the skydiver’s velocity can now be found by Marching Forward in Time.
Euler Method

\[ v(t + \Delta t) = v(t) + \frac{\Delta t}{m} \left( mg - \frac{1}{2} \rho [v(t)]^2 AC_D \right) \]

This equation can be cast into a form appropriate for solution using MATLAB. This is called the **Difference Equation**:  

\[ v_{k+1} = v_k + \frac{\Delta t}{m} \left( mg - \frac{1}{2} \rho (v_k)^2 AC_D \right) \]

Find the position by integrating the velocity:

\[ x_{k+1} = x_k + \frac{\Delta t}{2} (v_k + v_{k+1}) \]

Find the time by incrementing:

\[ t_{k+1} = t_k + \Delta t \]
Euler Method

Let $\Delta t = 0.1 \text{ sec}$, $C_D = 0.8$, $A = 0.4 \text{ m}^2$, $\rho = 1.225 \text{ kg/m}^3$, $m = 82 \text{ kg}$, $g = 9.81 \text{ m/s}^2$

Initial Velocity: $v(t = 0) = 0$ or $v(1) = 0$

Initial Position: $x(t = 0) = 0$ or $x(1) = 0$

$$v_{k+1} = v_k + \frac{\Delta t}{m} \left( mg - \frac{1}{2} \rho (v_k)^2 AC_D \right)$$

$$v(2) = (0) + \frac{(0.1)}{(82)} \left( (82)(9.81) - \frac{1}{2} (1.225)(0)^2 (0.4)(0.8) \right)$$

$$= 0.981 \text{ m/s}$$

$$x(2) = (0) + \frac{(0.1)}{2} [(0) + (0.981)] = 0.04905 \text{ m}$$

$$t(2) = 0 + 0.1 = 0.1 \text{ sec}$$
% Falling Skydiver: Euler Method
CD = 0.8; % Coefficient of Drag of the Skydiver's Body (Dimensionless)
A = 0.4; % Projected Area of the Skydiver's Body, m^2
rho = 1.225; % Density of Air, kg/m^3
m = 82; % Mass of Skydiver, kg
g = 9.81; % Acceleration due to Gravity, m/s^2
N = 3;
delta_t = 0.1;
t(1) = 0; % Initial Time, s
x(1) = 0; % Initial Position, m
v(1) = 0; % Initial Velocity, m/s

for k = 1:N
    v(k+1) = v(k) + delta_t/m*(m*g - 0.5*rho*v(k)^2*A*CD);
    x(k+1) = x(k) + delta_t/2*(v(k) + v(k+1));
    t(k+1) = t(k) + delta_t;
end
Euler Method

Falling Skydiver

Velocity $v$ (m/s) and Position $x$ (m)

Time $t$ (sec)
Euler Method

Falling Skydiver

**Graph:**
- **Skydiver Velocity (m/s):**
- **Skydiver Position (m):**

**Time (seconds):**
- 0 to 35
The solution of the differential equation for the skydiver is dependent on the chosen time step $\Delta t$. As the time step size decreases, the solution curves begin to overlap. This is called **Time-Step Independence**. Conversely, if $\Delta t$ becomes too large, the solution can become **unstable**, as shown for $\Delta t = 5.0$ seconds.
MATLAB ODE Solvers

- The **Euler Method** uses a fixed time step size that we specify and control.
- MATLAB has **built-in ODE solvers** that use variable step sizes. This speeds up the solution time. However, you no longer have control of the time step size.

  - **ode45**: Combination of 4\textsuperscript{th}- and 5\textsuperscript{th}-order Runge-Kutta methods.
  - **ode15s**: Used when **ode45** has difficulty.

- Basic syntax:
  
  
  \[
  [t, y] = \texttt{ode45}(\texttt{@ydot}, \texttt{tspan}, \texttt{y0})
  \]
MATLAB ODE Solvers

\[ [t, \ y] = \text{ode45}(@ydot,tspan,y0) \]

@ydot: Handle of function file that describes ODE equation.
tspan: Starting and ending values of time \( t \)
\[ [t_0, t_{\text{final}}] \]
y0: Initial value of \( y(0) \)

Use MATLAB to compute and plot the solution of the following equation:

\[ 10 \frac{dy}{dt} + y = 20 + 7 \sin(2t) \quad y(0) = 15 \]
MATLAB ODE Solvers

\[ 10 \frac{dy}{dt} + y = 20 + 7 \sin(2t) \quad y(0) = 15 \]

\[ \dot{y} = -\frac{1}{10} y + \frac{20}{10} + \frac{7}{10} \sin(2t) = -0.1y + 2 + 0.7\sin(2t) \]

Function File:

```matlab
function [ ydot ] = equation( t, y )

% page 387, T9.3-1

ydot = -y/10 + 2 + 0.7*sin(2*t);

end
```
MATLAB ODE Solvers

\[ \dot{y} = -0.1y + 2 + 0.7 \sin(2t) \quad y(0) = 15 \]

Script File: use ode45

```matlab
% page 387, T9.3-1
clc
clear
[t,y] = ode45(@equation, [0 100], 15);
plot(t,y), xlabel('Time (sec)'), ylabel('y')
```
MATLAB ODE Solvers

\[ \dot{y} = -0.1y + 2 + 0.7 \sin(2t) \quad y(0) = 15 \]

ode45 does not resolve the solution well: Note the sharp peaks. Use ode15 to improve resolution.
MATLAB ODE Solvers

\[ \dot{y} = -0.1y + 2 + 0.7 \sin(2t) \quad y(0) = 15 \]

Script File: Use `ode15s`

```matlab
% page 387, T9.3-1
clc
clear
[t, y] = ode15s(@equation, [0 100], 15);
plot(t, y), xlabel('Time (sec)'), ylabel('y')
```
MATLAB ODE Solvers

\[ \dot{y} = -0.1y + 2 + 0.7 \sin(2t) \quad y(0) = 15 \]
Problem 9.22:
Using the Euler Method, find the solution of the equation

\[ 6 \dot{y} + y = f(t) \]

if \( f(t) = 0 \) for \( t < 0 \) and \( f(t) = 15 \) for \( t \geq 0 \). The initial condition is \( y(0) = 7 \).

The Exact Solution is obtained by using the Integrating Factor Method:

\[ y(t) = 7e^{-t/6} + 15\left(1 - e^{-t/6}\right) \]

Plot the Exact Solution and the Euler Method Solution on the same graph to prove that your solution is Time-Step Independent.
Problem 9.22:

![Graph showing a function $y(t)$ over time $t$.]
Problem 9.25: 
The equation of motion of a rocket-propelled sled is, from Newton’s Law,

\[ m \dot{v} = f - cv \]

where \( m = 1000 \) kg is the sled mass, \( f = 75,000 \) N for \( t > 0 \) is the rocket thrust, and \( c = 500 \) N-s/m is an air resistance coefficient. Suppose that the initial velocity is \( v(0) = 0 \). Using the Euler Method, determine the speed of the sled until \( t = 10 \) seconds. Plot the sled velocity versus time, and show that the solution is independent of time step size.