CHAPTER 6: FUNDAMENTALS OF CONVECTION

So far, we've studied heat conduction based on being provided values of the convective heat transfer coefficient. The objective of this chapter is to begin to understand the mechanisms that control $h$, and to provide basic analyses to be able to estimate $h$.

- Velocity and thermal boundary layers
- Laminar vs. turbulent flows
- Continuity equation
- Momentum equations
- Energy equation
- Flat-plate flow
- Chilton--Colburn analogy between friction and convection coefficients.

**Velocity and thermal boundary layers**

Convection heat transfer is controlled by boundary layers at the surface of the solid.

$\delta(x) = \text{Hydrodynamic boundary layer thickness}$
\[ \delta(x) = \text{THERMAL BOUNDARY LAYER THICKNESS} \]

\[ T_S > T_{\infty} \quad \text{(HEATED PLATE)} \]

**Define B.L. Thicknesses:**

\[ \delta(x) \text{: occurs where } \frac{U}{U_{\infty}} = 0.99 \]

\[ \delta T(x) \text{: occurs where } \frac{T_S - T}{T_S - T_{\infty}} = 0.99 \]

**Important Surface Parameters:** \( \theta, C_f \) and \( h \)

**Hydrodynamics:**

\[ \tau_w = \mu \frac{\partial U}{\partial y} \quad \text{at } y = 0 \quad \text{VISCOUS SHEAR STRESS} \]

\( \text{NEWTON'S LAW OF VISCOSITY} \)

\[ C_f = \frac{\tau_w}{2 \rho V_{\infty}^2} \quad \text{COEFFICIENT OF FRICTION} \]

\[ F_D = C_f \cdot A \cdot \frac{8 V_{\infty}^2}{2} \quad \text{VISCOS DRAG FORCE} \]
THERMAL:

\[ \dot{q} = -k \frac{dT}{dy} \bigg|_{y=0} \]  

HEAT FLUX (FOURIER'S LAW)

\[ h = \frac{\dot{q}}{T_5 - T_0} \]  

COEFFICIENT OF CONVECTIVE HEAT TRANSFER (NEWTON'S LAW OF COOLING)

PHYSICAL SIMILARITY BETWEEN THE HYDRO B.L. AND THE THERMAL B.L. WILL BE USED LATER IN THE CHAPTER TO RELATE \( C_f \) AND \( h \).

LAMINAR VS. TURBULENT FLOWS

LAMINAR: SMOOTH STREAMLINES

TURBULENT: THREE-DIMENSIONAL EDDIES WITHIN THE OVERALL FLOW

TRANSITION: IN-BETWEEN LAMINAR AND TURBULENT

For flow over a flat plate, transition begins at

\[ Re_{tr} = \frac{V_{in} \times L}{\nu} = \frac{3 V_{in} \times L}{\mu} \approx 5 \times 10^5 \]
WHERE DOES TRANSITION OCCUR ON THE HOOD OF YOUR CAR WHEN TRAVELING AT 60 MPH?

\[ X_{TR} = \frac{Re_{TR}}{V_0} = \frac{\left(5 \times 10^5\right) \left(13.89 \times 10^{-6} \text{ m}^2\right)}{\left(60 \text{ mi/hr} \times \frac{1 \text{ hr}}{3600 \text{ s}} \times \frac{1609 \text{ m}}{1 \text{ mi}}\right)} = 0.296 \text{ m} = 0.974\]

**CONTINUITY EQUATION**

DERIVE CONSERVATION OF MASS EQUATION USING A DIFFERENTIAL CONTROL VOLUME:

\[ \left[ 9u + \frac{\partial}{\partial y} (9u)dy \right] \]

\[ \left[ 9u + \frac{\partial}{\partial x} (9u)dx \right] \]

\[ \sum \frac{\partial}{\partial x} v - \sum \frac{\partial}{\partial y} v = 0 \]

\[ 9u \cdot dy \cdot 1 + 9u dx - \int \left[ 9u + \frac{\partial}{\partial x} (9u)dx \right] dy \]

\[ + \int \left[ 9u + \frac{\partial}{\partial y} (9u)dy \right] dx \]

\[ \frac{\partial}{\partial x} (9u) + \frac{\partial}{\partial y} (9u) = 0 \]
**MOMENTUM EQUATIONS**

**NEWTON'S SECOND LAW:** \( \vec{F} = m \vec{a} \)

\( \vec{F} \) forces = net rate of linear momentum leaving C.V.

\( \vec{F} \) forces = body force + surface forces

Body forces = gravitational, magnetic, electro-magnetic, etc.

Surface forces = shear forces + normal forces

\[ \gamma_{xx} + \frac{\partial}{\partial y} (\gamma_{xy}) dx \]

\[ \gamma_{xy} + \frac{\partial}{\partial x} (\gamma_{yx}) dy \]

\[ \sigma_{xx} + \frac{\partial}{\partial x} (\sigma_{xx}) dx \]

\( \sigma_{xx} \) = normal stress in x-direction

\( \gamma_{xy} \) = shear stress in y-direction

\( \gamma_{xx} \) is not the same as static pressure \( P \):

\( \gamma_{xx} \to 0 \) for no flow, but \( P \) does not
\[ \sum \text{FORCES IN } X-\text{DIRECTION:} \]
\[ F_X = \left( \frac{\partial \sigma_{xx}}{\partial x} - \frac{\partial p}{\partial x} + \frac{\partial E_{xy}}{\partial y} \right) dx \, dy \]

\[ \sum \text{FORCES IN } Y-\text{DIRECTION:} \]
\[ F_y = \left( \frac{\partial E_{xy}}{\partial x} - \frac{\partial p}{\partial y} + \frac{\partial \sigma_{yy}}{\partial y} \right) dx \, dy \]

**X-DIRECTION MOMENTUM FLUX:**

\[ (\delta u)u + \frac{\partial}{\partial y} \left[ (\delta u)u \right] dy \]

\[ (\delta u)u \rightarrow (\delta u)u \]

\[ (\delta u)u \rightarrow (\delta u)u + \frac{\partial}{\partial x} \left[ (\delta u)u \right] dx \]

**X-DIRECTION MASS FLUX = \delta u**

**X-DIRECTION MOMENTUM FLUX DUE TO FLOW IN X-DIRECTION = \delta \delta u**

**Y-DIRECTION MASS FLUX = \delta v**

**X-DIRECTION MOMENTUM FLUX DUE TO FLOW IN Y-DIRECTION = \delta \delta v**
\[ \sum \textit{X-DIRECTION MOMENTUM FLUX:} \]
\[ M_x = \frac{\partial (\mathbf{u} u_y)}{\partial x} \, dx \, dy + \frac{\partial (\mathbf{u} u_y)}{\partial y} \, dy \, dx \]

\[ \sum \textit{Y-DIRECTION MOMENTUM FLUX:} \]
\[ M_y = \frac{\partial (\mathbf{u} u_x)}{\partial x} \, dx \, dy + \frac{\partial (\mathbf{u} u_x)}{\partial y} \, dy \, dx \]

\textit{NEWTON'S 2ND LAW IN X-DIRECTION:}
\[ \frac{\partial (\mathbf{u} u_y)}{\partial x} + \frac{\partial (\mathbf{u} u_y)}{\partial y} = \frac{\partial \mathbf{u}}{\partial x} \cdot \frac{\partial p}{\partial x} + \frac{\partial \mathbf{u}}{\partial y} \cdot \frac{\partial \mathbf{u}}{\partial y} + \mathbf{r} \]

\[ \text{LHS} = \frac{\partial}{\partial x}[\mathbf{u} u_y] + \frac{\partial}{\partial y}[\mathbf{u} u_y] \]
\[ = (\mathbf{u} \frac{\partial u}{\partial x}) + \mathbf{u} \cdot \frac{\partial (\mathbf{u} u_y)}{\partial x} + (\mathbf{u} \frac{\partial u}{\partial y}) + \mathbf{u} \cdot \frac{\partial (\mathbf{u} u_y)}{\partial y} \]
\[ = \mathbf{u} \left[ \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \right] + \mathbf{u} \cdot \left[ \frac{\partial (\mathbf{u} u_y)}{\partial x} + \frac{\partial (\mathbf{u} u_y)}{\partial y} \right] \]
\[ \text{A NEWTONIAN FLUID IS ONE IN WHICH SHEAR STRESS IS LINEARLY PROPORTIONAL TO THE RATE OF ANGULAR DEFORMATION:} \]
\[ \nabla_x = 2 \mu \frac{\partial u}{\partial x} - \frac{2}{3} \mu \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \]

\[ \nabla_y = 2 \mu \frac{\partial v}{\partial y} - \frac{2}{3} \mu \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \]

\[ I_{xy} = I_{yx} = \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \]

Substituting and taking derivatives gives

\[ 9 \left( 4 \frac{\partial u}{\partial x} + 5 \frac{\partial u}{\partial y} \right) = \mu \frac{\partial^2 u}{\partial y^2} - \frac{2 \rho}{\partial x} + X \]

Similar expressions for y-direction and z-direction momentum equations.

Energy Equation

\[ \nabla_p \left( \frac{1}{2} \frac{\partial T}{\partial x} + \frac{1}{2} \frac{\partial T}{\partial y} \right) = \frac{1}{2} \frac{\partial}{\partial x} \left( k \frac{\partial T}{\partial x} \right) + \frac{1}{2} \frac{\partial}{\partial y} \left( k \frac{\partial T}{\partial y} \right) + \mu \frac{\partial ^2 T}{\partial x \partial y} + \epsilon_{\text{gen}} \]

LHS = Net energy brought into C.V. due to fluid flow

\[ \Phi = \text{Volumetric heat generation due to viscosity} \]

\[ (\text{Viscous dissipation}) \]
FLAT-PLATE FLOW

Assume incompressible, constant properties, negligible body forces, no energy generation.

ORDER-OF-MAGNITUDE ANALYSIS GIVES:

\[ \delta : \begin{cases} \delta \ll \delta t \\ \frac{\partial u}{\partial y} \gg \frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \nabla \cdot \mathbf{u} \end{cases} \]

\[ \delta t : \frac{\partial T}{\partial y} \gg \frac{\partial T}{\partial x} \]

Continuity:

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \]

X-DIRECTION MOMENTUM:

\[ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial P}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} \]

Y-DIRECTION MOMENTUM:

\[ \frac{\partial P}{\partial y} = 0 \]
ENERGY:

\[ u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} + \frac{\nu}{C_P} \left( \frac{\partial u}{\partial y} \right)^2 \]

AFTER NON DIMENSIONALIZING, THE FOLLOWING IMPORTANT PARAMETERS APPEAR:

\[ Re = \frac{VL}{\nu} \text{ REYNOLDS NUMBER} \]

\[ Pr = \frac{\nu}{\alpha} \text{ PRANDTL NUMBER} \]

\[ Nu = \frac{HL}{k_f} \text{ NUSSELT NUMBER} \]

CONSERVATION EQUATIONS CAN BE SOLVED NUMERICALLY FOR FLAT-PLATE FLOW:

\[ \delta = 4.91 \cdot \frac{X}{\sqrt{Re_x}} \]

\[ Cf = 0.664 \cdot Re_x^{-1/2} \]

\[ \delta_T = 4.91 \cdot \frac{X}{Pr^{1/3} \sqrt{Re_x}} \]

\[ Nu = 0.332 \cdot Pr^{1/3} \cdot Re_x^{1/2} \text{ \text{ Pr > 0.6}} \]

IN GENERAL, \[ Nu = C \cdot Re_L^m \cdot Pr^n \]
The similarity between the hydrodynamic B.L. and the thermal B.L. is used to relate $C_f$ and $h$:

Chilton-Colburn Analogy:

\[
\frac{C_f}{2} = St \cdot Pr^{2/3}
\]

Where the Stanton number is

\[
St = \frac{h}{\delta Cp V_{m}}
\]