Set Notation and Axioms of Probability

Memory Hints:
\( \cap \) looks like A for And
\( \cup \) looks like U for Union

Complement: \( \bar{\neg} = \bar{Y} = Y \)

Null: \( \emptyset \quad P(\emptyset) = 0 \)

Mutually Exclusive: \( A \cap B = \emptyset \) If A and B are mutually exclusive then \( P(A \cap B) = 0 \)

Commutative Law
\[
A \text{ AND } B = B \text{ AND } A
\]
\[
A \cap B = B \cap A
\]
\[
A \text{ OR } B = B \text{ OR } A
\]
\[
A \cup B = B \cup A
\]

Distributive Law
\[
(A \text{ AND } B) \text{ OR } C = (A \text{ OR } C) \text{ AND } (B \text{ OR } C)
\]
\[
(A \cap B) \cup C = (A \cup C) \cap (B \cup C)
\]
\[
(A \text{ OR } B) \text{ AND } C = (A \text{ AND } C) \text{ OR } (B \text{ AND } C)
\]
\[
(A \cup B) \cap C = (A \cap C) \cup (B \cap C)
\]
\[
\bar{B} = (A \text{ AND } B) \text{ OR } (\bar{A} \text{ AND } B)
\]
\[
\bar{B} = (A \cap B) \cup (\bar{A} \cap B)
\]

DeMorgan’s Law
\[
\bar{(A \text{ AND } B)} = \bar{A} \text{ OR } \bar{B}
\]
\[
\bar{(A \cap B)} = \bar{A} \cup \bar{B}
\]
\[
\bar{(A \text{ OR } B)} = \bar{A} \text{ AND } \bar{B}
\]
\[
\bar{(A \cup B)} = \bar{A} \cap \bar{B}
\]

Axioms of Probability
\[
P(A) + P(\bar{A}) = 1
\]
\[
P(A) = 1 - P(\bar{A})
\]
\[
0 < P(A) < 1
\]
\[
\sum P(A_i) = 1
\]
Joint Probabilities

Addition Rule
\[ P(A \cup B) = P(A) + P(B) - P(A \cap B) \]
If A and B are mutually exclusive then \( A \cap B = \emptyset \) and \( P(A \cap B) = 0 \)
therefore, if A and B are mutually exclusive then \( P(A \cup B) = P(A) + P(B) \)

Conditional Probability
The conditional probability of an event A given an event B, denoted \( P(A \mid B) \) is
\[ P(A \mid B) = \frac{P(A \cap B)}{P(B)} \]
The conditional probability of an event B given an event A, denoted \( P(B \mid A) \) is
\[ P(B \mid A) = \frac{P(B \cap A)}{P(A)} \]

Multiplication Rule
\[ P(A \cap B) = P(A) \cdot P(B) \]
\[ P(A \cap B) = P(A \mid B) \cdot P(B) \]
\[ P(B \cap A) = P(B \mid A) \cdot P(A) \]
\[ P(A \mid B) \cdot P(B) = P(B \mid A) \cdot P(A) \]

Bayes' Theorem
\[ P(A \mid B) = \frac{P(B \mid A) \cdot P(A)}{P(B)} \]

Total Probability Rule
\[ P(B) = P(B \cap A) + P(B \cap \overline{A}) = P(B \mid A) \cdot P(A) + P(B \mid \overline{A}) \cdot P(\overline{A}) \]

Joint Probabilities
Dependent (Without Replacement) \( P(A \cap B) = P(A) \cdot P(B \mid A) \)
Independent (With Replacement) \( P(A \cap B) = P(A) \cdot P(B) \)

Independence
Two events are independent if any one of the following equivalent statements is true.
\[ P(A \mid B) = P(A) \]
\[ P(B \mid A) = P(B) \]
\[ P(A \cap B) = P(A) \cdot P(B) \]
If any one of the above statements is true, then all of them are true.

Mutually Exclusive
If A and B are mutually exclusive then
\[ P(A \cap B) = 0 \]
\[ P(A \cup B) = P(A) + P(B) \]
Independent, Dependent, & Mutually Exclusive

Multiplication Rule

\[
\begin{align*}
P(A \cap B) &= P(B \cap A) \\
P(A \cap B) &= P(A | B) \times P(B) \\
P(B \cap A) &= P(B | A) \times P(A) \\
P(A | B) \times P(B) &= P(B | A) \times P(A)
\end{align*}
\]

Independent

\[
P(A \cap B) = P(A) \times P(B)
\]

Mutually Exclusive

\[
P(A \cap B) = 0
\]

Hence independent events cannot be mutually exclusive.

Dependent

\[
P(A \cap B) = P(B \cap A) = P(A) \times P(B|A) = P(B) \times P(A|B)
\]

where \( P(A) \neq P(A|B) \) and \( P(B) \neq P(B|A) \), then events \( A \) & \( B \) are dependent.

If neither conditional probability \( P(A|B) \) or \( P(B|A) \) equals zero, then events \( A \) & \( B \) are dependent, but not mutually exclusive.

In either conditional probability equals zero \( \{P(A|B) = 0 \) or \( P(B|A) = 0\} \), the events \( A \) & \( B \) are dependent and mutually exclusive.

Example: Probability Matrices for Independent, Dependent, & Mutually Exclusive

<table>
<thead>
<tr>
<th></th>
<th>( A )</th>
<th>( \bar{A} )</th>
<th>( \bar{B} )</th>
<th>( B )</th>
<th>( P(A \cap B) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( B )</td>
<td>0.12</td>
<td>0.28</td>
<td>0.40</td>
<td>0.30</td>
<td>0.12 = P(A) \times P(B) hence Independent</td>
</tr>
<tr>
<td>( \bar{B} )</td>
<td>0.18</td>
<td>0.42</td>
<td>0.60</td>
<td>0.30</td>
<td>0.70</td>
</tr>
</tbody>
</table>

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<tr>
<th></th>
<th>( A )</th>
<th>( \bar{A} )</th>
<th>( \bar{B} )</th>
<th>( B )</th>
<th>( P(A \cap B) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( B )</td>
<td>0.09</td>
<td>0.31</td>
<td>0.40</td>
<td>0.30</td>
<td>0.09 hence Dependent, but not Mutually Exclusive</td>
</tr>
<tr>
<td>( \bar{B} )</td>
<td>0.21</td>
<td>0.39</td>
<td>0.60</td>
<td>0.30</td>
<td>0.70</td>
</tr>
</tbody>
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<tr>
<th></th>
<th>( A )</th>
<th>( \bar{A} )</th>
<th>( \bar{B} )</th>
<th>( B )</th>
<th>( P(A \cap B) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( B )</td>
<td>0.00</td>
<td>0.40</td>
<td>0.40</td>
<td>0.30</td>
<td>0.00 = P(A) \times P(B) hence Dependent, and Mutually Exclusive</td>
</tr>
<tr>
<td>( \bar{B} )</td>
<td>0.30</td>
<td>0.30</td>
<td>0.60</td>
<td>0.30</td>
<td>0.70</td>
</tr>
</tbody>
</table>
Probability Distribution Functions

**Discrete Probability Functions**

Probability Mass Function $f(x)$

- $f(x_i) \geq 0$
- $\Sigma f(x_i) = 1$
- $f(x_i) = \text{Prob} \ (X = x_i)$

Cumulative Distribution Function $F(x)$

- $F(x) = P(X \leq x) = \Sigma f(x_i)$
- $0 \leq F(x) \leq 1$

**Mean & Variance**

- $\mu = E(x) = \Sigma x \cdot f(x)$
- $\sigma^2 = V(x) = E(x - \mu)^2$
- $\sigma^2 = \Sigma [(x - \mu)^2 \cdot f(x)] = \Sigma x^2 f(x) - \mu^2$

**Continuous Probability Functions**

Probability Density Function $f(x)$

- $f(x) \geq 0$
- $\int f(x) \ dx = 1$
- $\text{Prob} \ (a \leq X \leq b) = \int f(x) \ dx$
- $\text{Prob} \ (X = x) = 0$

Cumulative Distribution Function $F(x)$

- $F(x) = P(X \leq x) = \int f(x) \ dx$
- $0 \leq F(x) \leq 1$

**Mean & Variance**

- $\mu = E(x) = \int x \cdot f(x) \ dx$
- $\sigma^2 = V(x) = E(x - \mu)^2$
- $\sigma^2 = \int (x - \mu)^2 f(x) \ dx = \int x^2 f(x) \ dx - \mu^2$
Methods of Counting

Multiplication Rule
For a process or operation with a sequence of \( k \) steps or alternatives, the total number of ways to complete the process is \( n_1 \times n_2 \times n_3 \times \ldots \times n_k \)

Permutations
A permutation is an ordered sequence of the elements of a set.
The number of permutations of \( n \) different elements is \( n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1 \)

Example for the elements a, b, c, d:
\[
abcd, abdc, acbd, acdb, adbc, adcb, 
bcad, bcda, bdac, bdca, 
cabd, cadb, cbad, cbda, cdab, cdba, 
dabc, dacb, dbac, dbca, dcab, dcba
\]
For \( n = 4 \), \( n! = 4 \times 3 \times 2 \times 1 = 24 \)

The number of permutations of subsets of \( r \) elements from a set of \( n \) different elements is
\[
P_r^n = \frac{n!}{(n-r)!}
\]

Example the four letters a, b, c, d taken two at a time, the number of permutations is
\[
P_2^4 = \frac{4!}{(4-2)!} = \frac{4!}{2!} = \frac{4 \times 3 \times 2 \times 1}{2 \times 1} = 12 \quad ab, ac, ad, ba, bc, bd, ca, cb, cd, da, db, dc
\]

The number of permutations of \( n = n_1 + n_2 + \ldots + n_r \) objects of which \( n_1 \) are of one type, \( n_2 \) are of a second type, \ldots, and \( n_r \) of are an \( r^{th} \) type is
\[
\frac{n!}{n_1! \times n_2! \times \ldots \times n_r!}
\]

Example: three O's and two X's \( (n = 3 + 2 = 5) \)
Permutations \( = \frac{5!}{(3!) \times (2!)} = \frac{120}{6 \times 2} = 10 \)
OOOXX, OOXOX, OXOXX, OXOXO, OXXOO, XXOOO, XOXOO, XOXOO, XOXO, XOXOX

Combinations
A combination is an unordered subset of \( r \) elements from a set of \( n \) elements.
The number of combinations, subsets of size \( r \) that can be selected from a set of \( n \) elements, is
\[
C_r^n = \frac{n!}{r! \times (n-r)!}
\]

Example the four letters a, b, c, d taken two at a time, the number of combinations is
\[
C_2^4 = \frac{4!}{2! \times (4-2)!} = \frac{4!}{2! \times 2!} = 6 \quad ab, ac, ad, bc, bd, cd
\]
Bernoulli Processes and Binomial Distributions

Conditions:
1. n independent trials
2. Only two possible outcomes per trial ("success" or "failure")
3. Probability of success on any one trial is constant = p
4. The random variable X is the number of successes in n trials, such that

\[
p(X = x) = C^n_x p^x (1 - p)^{(n-x)} \quad \text{where} \quad C^n_x = \frac{n!}{x!(n-x)!}
\]

Mean: \( \mu = E(X) = np \) \quad Variance: \( \sigma^2 = V(X) = np(1 - p) \)

Poisson Processes and Poisson Distributions

Poisson Process - Deals with the number of occurrences per interval.

Examples
Number of phone calls per minute
Number of cars arriving at a toll both per hour
Number of failures per 1000 hours
Number of bumps along a road per mile
Number of flaws in fabric per square yard
Number of pieces of debris per cubic meter of sea water

Conditions
For any interval over the real numbers:
1. Assume the occurrences happen at random throughout the interval.
2. Partition the interval into subintervals such that,
   a. The probability of more than one occurrence per subinterval is zero.
   b. The probability of one occurrence in a subinterval is the same for all the other subintervals, and is proportional to the length of the subinterval.
   c. An occurrence in any one subinterval is independent of an occurrence in any of the other subintervals.
3. Then the random occurrences are said to be a Poisson Process, such that;

If the mean number of occurrences in the interval is \( \lambda > 0 \), then the random variable X, (where X equals the number of occurrences in the interval) has a Poisson Distribution with parameter \( \lambda \) such that;

\[
f(x) = \frac{e^{-\lambda} \lambda^x}{x!} \quad \text{for} \ x = 0, 1, 2, \ldots
\]

Mean: \( \mu = E(X) = \lambda \) \quad Variance: \( \sigma^2 = V(X) = \lambda \)