Using In-Situ Error Tracking For Mode Selection in Proper Orthogonal Decomposition Reduced Order Modelling

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science in Engineering

by

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ABSTRACT

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Proper Orthogonal Decomposition (POD) provides a method of analyzing data and/or creating a Reduced Order Model (ROM). In this thesis, POD is used to create a ROM comprised of basis functions onto which the governing equations of a Computational Fluid Dynamics (CFD) problem are projected via Galerkin’s method. The model is of reduced order since the basis functions span a space smaller than the space from which they were created. A full order simulation called the training period is conducted first to obtain data to be used for creating the ROM using POD. The dominant characteristics are extracted from the data using Karhunen-Loève analysis. Computational expense is recouped by applying the ROM to the original system with its design parameters varied from the training conditions.

The influence of the individual POD basis functions on the solution varies. Additionally, changes to the design parameters and boundary conditions of the system may affect the influence of the individual basis functions. These basis functions can be ranked and the less influential basis functions can be truncated from the ROM, reducing computational expense. There are several methods that can be used to rank the basis functions. The work for this thesis seeks to find a way of determining the minimum amount of necessary modes needed to achieve an accurate solution. The methods used in this thesis are: tracking normalized error induced by exclusion of basis functions, the contribution of individual basis functions to the solution and the residual value of the excluded or truncated basis functions as proposed by Dr. Joseph Slater.
List of Symbols

Chapter 3

$\{u^k\}$  Ensemble of scaler fields
$x$  Spacial variable
$\dot{\omega}$  First order state vector
$\omega$  State vector
$R(w; \alpha)$  Vector of non-linear functions
$\alpha$  Time independent system parameters
$r$  Reduced order state vector
$\hat{w}$  State vector in modal Coordinates
$\Phi$  Matrix of mode Shapes
$\phi$  Individual mode Shapes
$S$  Subspace of $\mathbb{R}_n$
$\tilde{R}$  Residual
$S$  Snapshot matrix
$u_i$  Individual snapshot
$A$  Symmetric eigenproblem matrix
$a_{ij}$  Individual element of A
$N$  Number of states in the system
$M$  Number of time steps or solution points
$V$  Eigenvector matrix
$v$  Individual eigenvector
$\Lambda$  Eigenvalue Matrix
$\lambda$  Individual eigenvalue
Chapter 4

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<th>Symbol</th>
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<tr>
<td>$y_s$</td>
<td>Bump profile function</td>
</tr>
<tr>
<td>$x$</td>
<td>Location on x-axis</td>
</tr>
<tr>
<td>$\delta(t)$</td>
<td>Time dependent bump amplitude constant</td>
</tr>
<tr>
<td>$\delta_1$</td>
<td>Time independent bump amplitude parameter</td>
</tr>
<tr>
<td>$\omega$</td>
<td>Bump oscillation frequency</td>
</tr>
<tr>
<td>$\beta$</td>
<td>Bump amplitude modulation parameter</td>
</tr>
<tr>
<td>$U_\infty$</td>
<td>Freestream fluid velocity</td>
</tr>
<tr>
<td>$i, j$</td>
<td>Indices in x,y directions, respectively</td>
</tr>
<tr>
<td>$\Delta_{wall}$</td>
<td>Minimum grid spacing at bump boundary</td>
</tr>
<tr>
<td>$\hat{U}$</td>
<td>Matrix of flow variables, set in curvilinear coordinates</td>
</tr>
<tr>
<td>$\hat{E}, \hat{F}$</td>
<td>Transformed flux arrays, set in curvilinear coordinates</td>
</tr>
<tr>
<td>$\xi_i, \eta_i$</td>
<td>Coordinate Transforms</td>
</tr>
<tr>
<td>$\rho$</td>
<td>Density</td>
</tr>
<tr>
<td>$u$</td>
<td>Velocity in x-direction</td>
</tr>
<tr>
<td>$v$</td>
<td>Velocity in y-direction</td>
</tr>
<tr>
<td>$E_t$</td>
<td>Energy</td>
</tr>
<tr>
<td>$\Delta t$</td>
<td>Time step size</td>
</tr>
<tr>
<td>$r$</td>
<td>Reduced order state vector</td>
</tr>
<tr>
<td>$\Phi$</td>
<td>Matrix of mode shapes</td>
</tr>
<tr>
<td>$\Lambda$</td>
<td>Eigenvalue matrix</td>
</tr>
<tr>
<td>$R(\Phi \hat{w}; \alpha)$</td>
<td>Vector of non-linear functions in modal coordinates</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>Time independent system parameters</td>
</tr>
<tr>
<td>$I$</td>
<td>Identity matrix</td>
</tr>
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Chapter 5

Φ  Matrix of mode Shapes

\tilde{R}  Residual

\hat{w}  State vector in modal Coordinates

R(\Phi \hat{w}; \alpha)  Vector of non-linear functions in modal coordinates

\alpha  Time independent system parameters

\epsilon  Normalized error

\epsilon_i  Individual normalized error

\dot{\epsilon}  First derivative of reduced order state vector

\bar{\Phi}  Matrix of mode shapes normalized by eigenvector

I  Identity matrix

\rho  Density

u  Velocity in x-direction

v  Velocity in y-direction

E_t  Energy

Chapter 6

S_{xx}(j\omega)  One sided spectral density

G_{xx}(j\omega)  Two sided spectral density

\omega  Frequency of spectral density function

X(j\omega)  Fourier transformed data

j  Imaginary number
Chapter 7

\[ \rho \quad \text{Density} \]
\[ u \quad \text{Velocity in x-direction} \]
\[ v \quad \text{Velocity in y-direction} \]
\[ E_t \quad \text{Energy} \]
\[ p \quad \text{pressure} \]
\[ \gamma \quad \text{Ratio of specific heats} \]
\[ M_\infty \quad \text{Freestream Mach number} \]
\[ y_s \quad \text{Bump profile function} \]
\[ x, y \quad \text{Spacial variables} \]
\[ t \quad \text{Time variable} \]
\[ \Delta t, \Delta t_r \quad \text{Time step size, reduced order time step size} \]
\[ \delta_1 \quad \text{Time independent bump amplitude parameter} \]
\[ \omega \quad \text{Bump oscillation frequency} \]
\[ \beta \quad \text{Bump amplitude modulation parameter} \]

Chapter 8

\[ \delta_1 \quad \text{Time independent bump amplitude parameter} \]
\[ \omega \quad \text{Bump oscillation frequency} \]
\[ \beta \quad \text{Bump amplitude modulation parameter} \]
\[ M_\infty \quad \text{Freestream Mach number} \]
\[ \Delta t_r \quad \text{Reduced order time step} \]
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Additional thanks and appreciation to Dr. Joseph Slater for his guidance.
Dedicated to
My family and Anheuser-Busch
Introduction

Proper Orthogonal Decomposition (POD) can be used as a method of analyzing data to obtain a low dimensional approximate description of a high dimensional process or data set. These low dimensional approximate descriptions are used as basis functions for a Reduced Order Model (ROM). The choice and number of basis functions is critical to the accuracy of the ROM. The choice of basis functions is established by the POD method. The number of basis functions in the ROM is based on the desired accuracy. The work performed in this thesis focuses on achieving quality solutions of a Computational Fluid Dynamics (CFD) problem while using reduced order models developed using the POD method. This is accomplished by investigating the influence on accuracy of the individual basis functions used in the reduced order model. This method is abbreviated POD/ROM.

In the work for this thesis, POD/ROM is applied to a numeric fluid flow solver. Fluid flow problems are solved using computer codes and digital computers. The methods used to solve large complex fluid flow problems are computationally expensive. This expense is compounded if the flow problem is subjected to solution under varied conditions (iterative design). Reduced order modelling offers a way of reducing the computational expense by eliminating the need for a full order solution but may come at the cost of solution accuracy or stability. The ROM uses basis functions onto which the governing equations of a fluid dynamics problem are projected. The system is then of reduced order since the space spanned by the ROM basis function is smaller than the space of the full order system. The method in which the POD/ROM is used in this work allows reduction in computational expense by the use of larger integration time steps. Larger time steps can be taken because the POD/ROM filters out the numerical instability that results from the larger time step. The amount of filtering is dependent on the number of modes. This is where the accuracy
of the POD/ROM method is affected.

The POD method is used for creating a reduced order model. This model is created from basis functions or vectors that are generated from data provided from the numerical solution of the full-order system (called the training period). Background on the POD method can be found in Section 3. Karhunen-Loève analysis is used to extract the POD modes or basis vectors from the discrete data taken during the training period. A description of Karhunen-Loève analysis can be found in Section 3.2. Some background information is provided about the fluid dynamics problem being studied. Chapter 4 provides a brief discussion of the panel flutter problem that is used in this work and the CFD code used to solve the problem. An extensive discussion of the fluid problem is not provided since the flow solver was developed prior to the beginning of this work [1]. Background on the flow solver can be found in [2]. No changes were made to the flow solver in this work. The operation of the code is described in Section 4.3. The methodology used to implement the POD/ROM in the code is found in Section 4.4.

It is desirable to be able to determine the ROM effectiveness without adding the computational expense of a full-order simulation. The effectiveness of the ROM is checked by the amount of relative error introduced into the solution. The accuracy of the reduced order model is tracked in-situ by checking relative error since this error does not require the full order solution to calculate. These error calculations are for the normalized error, normalized modal contributions and residual values of the POD modes. The background on these calculations are shown in Chapter 5. The boundary conditions, on and off-design parameters are given in Section 5.3.

The results from the simulations using the on-design conditions are given in Chapter 6. The full and reduced order solutions are obtained at these conditions. The full order solution serves the purpose of generating the data for creating the reduced order model. The reduced order simulation at on-design conditions establishes the baselines for the errors. Simulation results at off-design condition are given in Chapter 7. All simulations at these conditions are reduced order. Chapter 8 gives the results of the simulations using a larger time step with the reduced order model.

Different design variables have varying effect on the accuracy of the reduced order solution. Trend analysis is used to draw conclusions on ways to use in-situ error to determine model effective-
ness and accuracy by comparing these errors for the different design situations. Chapter 9 provides the conclusions drawn from this research. Chapter 10 provides a listing of the computer programs used throughout this thesis.
Since its inception, POD has been applied to a broad range of disciplines. Some of these disciplines are image processing and pattern recognition [3], oceanography [4], weather prediction [5], mechanical vibrations [6], and fluid mechanics [7, 8]. Fluid mechanics or fluid dynamics is the primary interest area of this research and where most of the effort was spent on literature review. According to Lumley [9], the idea of POD can be traced back to independent investigations by Kosambi [10] in 1943, Loève [11] in 1945, Karhunen [12] in (1946), Pougachev [13] in 1953 and Obukhov [14] in 1954. The beginnings of POD are rooted in statistical analysis. Lumley [15] was the first to apply the method to turbulence in 1967. Holmes, Lumley and Berkooz [8] have authored a book that is most informative on the application of proper orthogonal decomposition to fluid dynamics. This book provides a large amount of background information concerning the foundations of proper orthogonal decomposition, its optimality, uses and properties. The paper by Chatterjee [16] provides a brief overview of POD as well as a few simple numeric examples of its applications performed using MATLAB (with MATLAB code provided). The POD method used in [16] is Singular Value Decomposition (SVD). The method used in this research is the method of snapshots, developed by Sirovich [17]. The method of snapshots has become prevalent in most of the applications of POD in the analysis of dynamical systems.

Cizmas and Palacios [7] investigated the use of proper orthogonal decomposition in the numerical simulation of a rotor-stator interaction in a one-stage turbine. Method of snapshots was used to develop the basis functions. These functions were then used as a reduced order model via Galerkin’s projections. The error in the energy variable and the skin friction between the full and reduced-order model were used in the determination of the method’s accuracy. Errors of groups of
neglected modes were tracked (the error associated with individual modes was not tracked). The variable used in this paper for the error calculations was the conserved energy flow variable. This variable was also used for determining ROM validity and stability. This variable was chosen because it has the potential for the largest variance in value across the flow field. Additionally, it was calculated from all the other flow variables. Any error in them would be magnified in the calculation of the conserved energy.

Cazemier, Verstappen and Veldman [18] used proper orthogonal decomposition in a reduced order model of cavity flow. They used the method of snapshots to develop the basis function and created the snapshots from direct numerical simulations. The reduced order model was then applied to the driven cavity problem at Reynolds numbers different than the one at which snapshots were taken. It was found that a closure model was needed in the reduced order simulations above a certain Reynolds number. Illustrations of the first few modes projected onto the flow field give a visual reference to the characteristics contained in each of the first few modes. They show the coherent structures that are dominant in that particular mode very well. Mode selection was based on energy content, wherein modes were ranked and ordered by eigenvalue. Although the authors point out that neglecting a particular mode may result in the loss of the flow structures represented by that mode, there is no mention of how to tell if one mode may have more influence than another.

Anttonen, King and Beran [19] applied POD to a reduced order model with deforming grids. There is a difficulty in using proper orthogonal decomposition for reduced order modelling with moving grids. The POD method uses basis functions that are spatially dependent. When the snapshot is taken, the order that the variables are placed in a snapshot is not important but the index location between snapshots should match the same physical location that the variable’s value was derived. When the governing equations are projected on the basis functions for conditions that do not match the training scenario (i.e.: the grid has moved), there is a skew in the spatial relation between the basis functions and the governing equations, resulting in an erroneous solution. The authors found that the robustness of the reduced order model (ROM) created using POD could still be preserved if the reduced order model was applied to situations were the grid deformation was similar to the grid deformation present in the training period (snapshot selection period). It was also
found that more modes were required to be retained in order to achieve accuracy levels comparable to models without deforming grids. In order to increase the accuracy of the POD/ROM, the concept of multi-POD was introduced. The idea was to combine POD models from several different grid deformation scenarios, thus better equipping POD to account for grid motion. This paper contained no discussion on importance of individual modes. Modes were selected consecutively based on energy content. Equations for error calculations were given but no results for error were presented.

Feeny and Kappagantu [6] applied POD to vibration problems. The authors relate the normal modes found in vibrational systems to the modes found by proper orthogonal decomposition of the same system, lending insight to the differences as well as the similarities between the two. The less influential modes in vibrating systems are the higher modes which correspond to smaller eigenvalues (smaller eigenvalues corresponded to lower energy content in the previous works). The system used in the examples were sprung masses with no or light damping. It was shown that the proper orthogonal modes and the vibration modes converge. However, all of the modes for the discrete systems were retained in the reduced order simulations in these examples. Therefore no explanation for modal influence was given.

Hall, Thomas and Dowell [20] use a different approach in the development and application of their reduced order model. They developed their ROM in the frequency domain. Up to this point, the POD/ROMs discussed above have been developed in the time domain. In this work, the snapshots were taken at multiple discrete frequencies instead of at discrete instants in time. The application of the model was still the same. The ROM was applied to a small-disturbance about a 2-D airfoil. The use of error in determining modal importance was not discussed.

Rowley, Colonius and Murray [21] used POD in the simulation of a compressible fluid flow. The fluid flows in the previous literature reviews have been assumed incompressible. Flow was analyzed over a flat surface with a square cavity in it. The POD method was used to extract basis functions from snapshots taken by full-order direct numerical simulations. More thermodynamic values need to be retained since a divergence-free condition in the velocity field no longer exists when the flow field goes from incompressible to compressible flow. The projections of the first few modes on the flow field are shown, along with a time trace of the time dependent Galerkin
coefficients. No explanation is given for mode selection or error induced by neglecting modes.

Two papers provided much of the background for the research presented here. The first is by Mortara, Slater, and Beran [22] and applies POD to the analysis of non-linear panel flutter. The panel was subjected to a supersonic inviscid flow. The full and reduced order numerical simulations were carried out in MATLAB and the results were compared to the problem solved by Dowell in 1966. The idea of separating the POD basis functions into retained and truncated modes was introduced and used in the reduced order modelling process. The second paper is by Slater, Pettit and Beran [2]. The response of a transonic fluid moving above a panel subjected to a forced localized oscillation was simulated. POD was used to create a reduced order model with which the original system was simulated using the ROM at and away from the conditions used to create the ROM. Both works separate the basis functions generated by the POD method into retained and truncated modes. The retained modes were used in the ROM and the truncated modes were tracked. Eigenvalues were used to establish the criteria for truncation. The work in [2] showed the potential importance of the truncated modes through the calculation of a residual and the use of residual values to determine basis functions for the ROM in-situ. These works also show the periodic nature of the basis functions in the reduced order model which were a result of a periodic forcing function.

Potential destabilization of a reduced order model using proper orthogonal decomposition was discussed in the paper by Slater [23]. A demonstration was given utilizing MATLAB code of how the choice of basis functions used in the ROM created by proper orthogonal decomposition can affect the stability of a reduced order model. Keeping an increasing number of modes in the model caused an instability in the solution and was demonstrated in the example. However, the solution was made stable again by including more modes than the number of modes making the problem unstable. One possible cause of the instability was the oscillatory or periodic behavior of the modes. This suggests that some of the neglected modes were carrying information important to the stability of the solution. A suggestion of measuring the importance of the individual modes by error tracking was made.
Proper Orthogonal Decomposition

The proper orthogonal decomposition process is also known under the aliases Karhunen-Loève (K-L) decomposition, Karhunen-Loève expansion, eigenfunction analysis, empirical orthogonal eigenfunctions, and principal component analysis. The capacity in which proper orthogonal decomposition is used here is to capture the dominant structures in a fluid flow. These structures are characteristic recurrent forms called coherent structures [8]. Coherent structures are energetically dominant and are what one sees when looking at a flow. Examples of coherent structures are wakes, eddies and vortices.

The POD method provides a method for extracting the proper orthogonal modes from experimental or simulated data. The proper orthogonal modes contain information about the coherent structures in the flow and carry the greatest kinetic energy on average [8]. These modes are used to construct a set of basis functions (continuous system) or basis vectors (discrete system). The basis functions or basis vectors form the reduced order model. The modes of the reduced order model form low dimensional subspaces onto which the governing equations are projected. This projection is accomplished by Galerkin’s method.

It is important to realize that proper orthogonal decomposition is a linear process. POD is commonly used in non-linear situations. The nested sequence of subspaces rendered by the basis functions from the POD process are linear spaces, even though the source data or dynamic systems may be non-linear. This is one of the method’s strengths as well as a limitation. Its strength is that it captures the coherent structures from that system by using data from the system or the system itself. A weakness of the method is its sensitivity to changes to system parameters of the original system from which the POD modes are based. This weakness can be magnified by changes to parameters.
3.1 Method of Snapshots

The method of snapshots for use with POD was first presented by Sirovich [17, 24, 25] and is a popular method for finding ROM basis functions. This procedure allows a faster numerical solution of the eigenvalue problem used to generate the proper orthogonal modes by creating a matrix of lower order than the full-order system. This is only possible in the case of $N \gg M$, where $N$ is the number of degrees of freedom in the system and $M$ is the number of ensemble functions or snapshots deemed necessary to represent the system. The potential of this method is readily apparent for high-order systems. The method of snapshots is based on the N-sized data vectors $\{u_i\}_{i=1}^M$ and eigenvectors $\Phi_k$ spanning the same linear space [8, 17]. This implies that the eigenvectors can be written as a linear combination of the data vectors

$$\Phi_k = \sum_{i=1}^{M} \nu_k^i u_i, \quad k = 1, ..., M \quad (3.1)$$

After substituting (3.1) in the eigenvalue problem $R(x, y)\Phi(y) = \lambda \Phi(y)$, the coefficients $\nu_k^i$ are obtained by solving the eigenvalue problem

$$A\nu = \lambda \nu \quad (3.2)$$

where $\nu_k^i$ is the $k^{th}$ eigenvector of (3.2). The matrix $A$ is the symmetric $M \times M$ matrix defined as $[a_{ij}] = (1/M)(u_i, u_j)$. $(\cdot, \cdot)$ represents the standard vector inner product, i.e. $(u_i, u_j) = u(x_1, t_i)u(x_1, t_j) + ... + u(x_M, t_i)u(x_M, t_j)$. This allows the eigenvectors of the $N \times N$ matrix $R$ to be computed from the eigenvectors of the $M \times M$ matrix $A$. The data vectors $\{u_i\}_{i=1}^M$ (snapshots) are arranged in a column matrix $S = [u_1 \, u_2 \, ... \, u_M]$ called the snapshot matrix. The matrix $A$ is the product of $A = S^T S$. This is the matrix that the Karhunen-Loève proper orthogonal
decomposition method is applied to in finding the basis vectors for the reduced order model. This method can also be seen as a Singular Value Decomposition (SVD). Reference [16] illustrates how POD used in this manner is a special case of a SVD.

### 3.2 Karhunen-Loève (K-L) Analysis

Karhunen-Loève analysis is used to find the mode vectors \( \Phi = \{\phi\}_i^M \). A set of snapshot vectors, \( u_i \), are generated from time integration of the full order system equations (3.7). The hope is that the space covered by the basis functions vectors generated from data gathered in the training simulation is sufficient to cover the space within which subsequent reduced-order simulations will operate. If these snapshots capture enough of the dominant features or coherent structures of the system during the training period, the reduced order system should be robust to variations of the system’s time-invariant parameters. Simulation conditions different than those of the training period are referred to as off-training or off-design conditions. Generally, the parameters \( \alpha \) in equation (3.7) are varied from the training condition in the reduced order simulations (off-training simulations).

The snapshot matrix \( S \) is formed from data sampled at pre-determined points in time. The time between gathering of snapshots vectors \( u_i \) can be made adaptive but remains fixed in this research. If \( \nu \) in equation (3.1) is considered as an eigenvector of the matrix \( A = S^T S \), equation (3.1) can be rewritten in matrix form, \( \Phi = SV \). Substituting \( \Phi = SV \) into the term \( (\Phi^T \Phi) \) in equation (3.14) yields

\[
(\Phi^T \Phi) = V^T S^T S V \tag{3.3}
\]

Since \( V \) was chosen to be the matrix of eigenvectors of \( A \),

\[
(\Phi^T \Phi) = V^T S^T S V = \Lambda \tag{3.4}
\]

where \( \Lambda \) is the diagonal eigenvalue matrix of \( A \) and \( V \) is orthonormal, \( V^T = V^{-1} \). The solution of the eigenvalue problem

\[
AV = V\Lambda \tag{3.5}
\]
along with the relation $\Phi = SV$ yields a set of orthogonal basis vectors $\Phi$.

The eigenvalues are used for initial selection of modes. The magnitude of each of the eigenvalues in $\Lambda$ provides an indication of the amount of so-called energy content of its particular associated eigenvector. This may suggest how significant the eigenvector might be to the physics of the problem. This so-called energy is not necessarily related to a physical energy. In the case of velocity measurements, these eigenvalues are related to the kinetic energy of the fluid [16]. The more commonly used form of the reduced-order equations is obtained by substituting $\Phi = SV$ and equation (3.4) into equation (3.14) yielding

$$\dot{\hat{w}} = (\Lambda)^{-1}V^T S R(\Phi \hat{w}, \alpha) \quad (3.6)$$

The shortcoming of the K-L Analysis approach is that there is no guarantee that the basis vectors obtained from one simulation will be sufficient to span the space of another simulation, although it has been shown in several cases that POD is robust to varying parameters [2, 8, 18]. It becomes necessary to truncate the vectors $\Phi$ to only those containing sufficient energy content (significant eigenvalues, $\lambda$) [1] when performing the reduced-order time integration or else equation (3.6) can break down due to the near singularity of the matrix $\Lambda$. This truncation is generally done during the Karhunen-Loève process after the $\Phi$ matrix has been determined. Further truncations (or additions) may be made during the reduced order modelling simulations and are discussed in Section 4.4.

### 3.3 Galerkin Projection

Galerkin’s method can be used to construct a numerical solution to partial differential equations or create a mathematical model of a dynamical system. This model is generally of lower order than the original system which it represents. For instance, when fluid flow is simulated, integration is used to find the solution to a set of governing differential equations in time and space. Galerkin projection provides a method for converting the infinite dimensional set of governing equations to
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a finite set of difference equations. Functions representing the governing equations are projected onto a finite k-dimensional subspace $S \subset \mathbb{R}^n$ of the full state space of the system when applying Galerkin’s method. This allows for faster solution when the size of $S$ is less than $\mathbb{R}^n$. The basis vectors used in the projections form an orthogonal basis of the subspace $S$. Reference [8] provides proof that POD provides an optimal basis.

To demonstrate the application of Galerkin projection, consider a set of nonlinear first order state space equations representing a dynamical system evolving in a Hilbert space $H$ of the form

$$\dot{w} = R(w; \alpha) \quad (3.7)$$

where $R$ is a vector of nonlinear functions and $\alpha$ is a parameter or list of parameters upon which the system depends, independent of the system states. Assume a (potentially reduced order) solution to equation (3.7) of the form

$$r(t) = \sum_{n=1}^{N} \hat{w}_n(t) \phi_n \quad (3.8)$$

where $\hat{w}_n(t)$ represent so-called modal coordinates, $\phi_n$ represent so-called mode shapes and $r(t) \in S$ is the approximation to $w(t)$. The mode shapes $\{\phi_n \in H \mid n = 1, \ldots, N\}$ form an orthonormal basis for the subspace $S$. The solution given by equation (3.8) can also be written in matrix form

$$r(t) = \Phi \hat{w}(t) \quad (3.9)$$

where $\Phi = [\phi_1, \phi_2, \phi_3, \ldots, \phi_n]$ and $\Phi : H \rightarrow S$. Substituting (3.9) into the equation of motion (3.7) yields

$$\Phi \dot{\hat{w}} = R(\Phi \hat{w}, \alpha) \quad (3.10)$$

The residual $\tilde{R}$ is then defined to be the difference between the left and right hand sides of equation (3.10)

$$\tilde{R} = \Phi \dot{\hat{w}} - R(\Phi \hat{w}, \alpha) \quad (3.11)$$

The weighted residual produced by the $n^{th}$ so-called mode shape $\phi_n$ is then found by pre-multiplying
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Putting the weighted residuals in vector form and constraining them to be equal to zero yields

\[
\Phi^T \tilde{R} = \Phi^T \left( \Phi \dot{\hat{w}} - R(\Phi \hat{w}, \alpha) \right) = 0
\]  
(3.13)

Solving for \( \dot{\hat{w}} \) yields the reduced-order equations of motion

\[
\dot{\hat{w}} = (\Phi^T \Phi)^{-1} \Phi^T R(\Phi \hat{w}, \alpha)
\]  
(3.14)

To convert the solution back into computational space, equation (3.14) is pre-multiplied by \( \Phi \)

\[
\Phi \dot{\hat{w}} = \Phi \left( \Phi^T \Phi \right)^{-1} \Phi^T R(\Phi \hat{w}, \alpha)
\]  
(3.15)

Using equation (3.9)

\[
\dot{r} = \Phi \left( \Phi^T \Phi \right)^{-1} \Phi^T R(\Phi \hat{w}, \alpha)
\]  
(3.16)

which can be numerically integrated in time using larger time steps [22] than the original equations of motion. The results achieved will not be identical to the full-order simulation since this is a reduced order method. However, these results should be functionally equivalent to those of the full-order simulation.
The CFD code used in the solution of the fluid flow in the current problem is called RAPOD. This code was developed by Dr. Philip S. Beran and Dr. Chris Pettit [1]. RAPOD was developed to be problem independent, meaning that the grid generation and system parameter inputs are independent of the RAPOD flow solver. The method of implementation of the reduced order modelling was changed and error tracking routines were changed or added. These changes are discussed in Section 4.4 and Chapter 5.

4.1 Test Problem

The test problem is the unsteady response of an inviscid flow field to time dependent geometric changes of a 2-D oscillating bump in an infinite segmented panel. The panel lies in the $y = 0$ coordinate plane. The surface deformation of the segments described by $y_s(x, t)$ lies between $x = -\frac{1}{2}$ and $x = \frac{1}{2}$ on the panel surface. The values of $x$ at these locations were chosen so that the deforming segment length $c$ normalizes to unity. The remainder of the panel is defined as flat away from the deforming segment, $(y_s(x, t) = 0 \mid |x| > \frac{1}{2})$. The time dependent geometry of the bump is given by

$$y_s(x, t) = \delta(t) \left(1 - 4x^2\right) \quad (|x| \leq \frac{1}{2}), \quad (4.1)$$

$$\delta(t) = \delta_1 \cos(\omega t) \left(1 - e^{-\beta t}\right), \quad (4.2)$$

where $\delta_1$ is the maximum bump deflection amplitude, $\omega$ is the bump oscillation frequency and $\beta$ is the modulation parameter used to adjust the short-time or start-up behavior of the system. An
exaggerated illustration of this bump is shown in figure (4.1).

![Schematic of panel and coordinate system.](image)

Figure 4.1: Schematic of panel and coordinate system.

The spatial and temporal variables describing the bump motion were non-dimensionalized using length \( c \) and the aerodynamic characteristic time based on the far-field velocity \( U_\infty \). This results in the deforming segment taking the shape of a parabolic arc with a maximum deflection at \( x = 0 \) of \( |\delta(t)| \) with a periodic motion governed by the cosine function as the exponential behavior decays to zero, giving the motion a transient behavior at the beginning of the simulation.

4.2 Grid Construction

The flow is simulated over a panel lying in a physical domain of length \( L \) and height \( H \) centered about \( x = 0 \). The domain is discretized using \( I \) nodes in the stream-wise (parallel to panel) direction and \( J \) nodes normal to the panel. Grid points are denoted by the indices \( i \) \((1 \leq i \leq I)\) and \( j \) \((1 \leq j \leq J)\). The flow variables are evaluated at these grid points. Grid points corresponding to \( j = 1 \) are placed on the \( x \)-coordinate line and do not move with changes in \( \delta \). Grid points are clustered in the direction normal to the panel at the panel surface, with the minimum spacing denoted by \( \Delta_{\text{wall}} \). Spacing of grid points grows geometrically normal to and away from the panel boundary. In the stream-wise direction, the node spacing is chosen to be uniform over the deforming panel segment, while growing geometrically upstream of the leading edge (LE) (positioned at \( i = I_{LE} \)) and downstream of the trailing edge (TE) (positioned at \( i = I_{TE} \)). A baseline grid is shown in figure (4.2) and is constructed with the following values: \( I = 141, J = 71, L = 15, H = 6, I_{LE} = 55 \) and \( I_{TE} = 85 \). For the baseline grid, \( \Delta_{\text{wall}} \approx 0.0137 \).
4.3 Method of Solution

The governing equations of the flow being studied are partial differential equations. The solution of these equations is conducted numerically using a first order spatial accuracy method. The governing equations are discretized and marched forward in time to render the solution to the flow. This process applies equally to both the full and reduced order numerical simulations.

The governing aerodynamic equations of the flow are the Euler equations. These equations are cast in a non-dimensional form and set in a curvilinear coordinate system $(\xi, \eta)$, with $\xi$ representing the $x$-coordinate and $\eta$ representing the $y$-coordinate.

$$\frac{\partial \hat{U}}{\partial t} + \frac{\partial \hat{E}(U)}{\partial \xi} + \frac{\partial \hat{F}(U)}{\partial \eta} = 0,$$

where $U \equiv [\rho, \rho u, \rho v, E_t]^T$ is the vector array of conserved variables, $\hat{U} = U/(\xi_\eta \eta_x - \xi_y \eta_x)$ is the
vector array of conserved variables transformed by the Jacobian \((\xi_x \eta_y - \xi_y \eta_x)\) [26]. The variables \(\hat{E}\) and \(\hat{F}\) are transformed flux arrays. The governing equations are discretized using the upwind total variation diminishing (TVD) scheme of Harten-Yee [27, 28]. The discretization of second-order or first-order spatial accuracy is expressed as

\[
\frac{\hat{U}_{i,j}^{n+1} - \hat{U}_{i,j}^n}{\Delta t} = \tilde{\hat{F}}_{i,j} - \tilde{\hat{F}}_{i,j-\frac{1}{2}} + \tilde{\hat{E}}_{i-\frac{1}{2},j} - \tilde{\hat{E}}_{i+\frac{1}{2},j},
\]

where the arrays \(\tilde{\hat{E}}\) and \(\tilde{\hat{F}}\) are modified numerical fluxes that implement the TVD formulation.

## 4.4 Implementation of POD

The POD-ROM model is implemented in the following way. First, a training simulation is run. This simulation uses the discretized full-system equations to integrate the flow in time. Snapshots are taken of the flow at certain predetermined discrete time intervals. The time interval between two successive snapshots should be large enough for the snapshots to be almost uncorrelated. Having snapshots that are as uncorrelated as possible results in a more well-posed eigenproblem and better basis functions. Snapshots are column vectors containing the values of the system’s flow variables at the instant in time when the snapshot is taken.

The snapshot vectors are placed in the columns of the snapshot matrix, \(S\). The order in which the flow variables are placed in the individual snapshot vectors is not important. However, it is important that the index position from snapshot to snapshot to which a grid point variable is written remains consistent. Loss of this consistency in the snapshot matrix would result in an invalid solution.

Karhunen-Loève (K-L) Analysis is performed on the snapshot matrix \(S\) using the procedure set forth in Section 3.2 after the training simulation is conducted. This analysis yields the basis vectors \(\Phi\) that are used in the ROM. Using equation (3.4), equation (3.16) can be written

\[
\dot{\mathbf{r}} = \Phi (\Lambda)^{-1} \Phi^T \mathbf{R}(\Phi \hat{\mathbf{w}}; \alpha) \tag{4.5}
\]
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The $\Phi$ matrix is normalized by pre-multiplying by the inverse square root of the eigenvector matrix $\Lambda$

$$\bar{\Phi} = \Phi \Lambda^{-\frac{1}{2}}$$  \hspace{1cm} (4.6)

which has the property

$$\Lambda^{-\frac{1}{2}} \Phi^T \Phi \Lambda^{-\frac{1}{2}} = \Lambda^{-\frac{1}{2}} \Lambda \Lambda^{-\frac{1}{2}} = \bar{\Phi}^T \bar{\Phi} = I$$  \hspace{1cm} (4.7)

Using the simplified form of $\bar{\Phi}$, equation (4.5) now becomes

$$\dot{\mathbf{r}} = \bar{\Phi} \bar{\Phi}^T \mathbf{R}(\bar{\Phi} \mathbf{\bar{w}}; \alpha)$$  \hspace{1cm} (4.8)

Application of K-L Analysis and Method of Snapshots renders a set of basis vectors $\bar{\Phi}$ equal to the number of snapshots taken. Not all of these basis vectors are used in the subsequent reduced order model. The basis vectors with the correspondingly highest eigenvalues are retained. Further paring of these may be done as accuracy of the reduced order solution allows.

The matrix $\bar{\Phi}$ needs to be calculated only once since $\Phi$ and $\Lambda$ are independent of time and off-training conditions. When reduced order simulations are conducted using off-training conditions, the ROM matrix $\bar{\Phi}$ only needs to be read in once prior to a simulation or deck of simulations.
Numerical Simulation and Error Calculation

5.1 Error Tracking

The error induced by the reduced order model may be monitored during the solution process in order to determine its effectiveness. The error may be used to determine if the ROM should be updated or replaced if the error passes a predetermined threshold. Relative error tracking methods are used in the code due to their simplicity. These error calculations are performed during the verification and off-training reduced order simulations (called in-situ error tracking).

In-situ error tracking is implemented by calculating the normalized modal contributions, residual values and the normalized error. Residuals were used by Slater et al. [2] to determine mode significance. The residual values of the truncated modes are calculated by the use of equation (3.11). The matrix $\Phi$ in the residual calculation contains both the retained modes and truncated modes.

The normalized error was used to provide an indication of the error induced into the solution by the ROM at each time step. The normalized error was found in a manner similar to the residual error in the paper by Slater [2]. Let equation (4.8) be the filtered reduced order solution,

$$\dot{r}_f = \tilde{\Phi}\tilde{\Phi}^T R(\tilde{\Phi}\tilde{\omega})$$

(5.1)
and the non-filtered reduced order solution is given by

$$\dot{\bar{r}} = R(\Phi \hat{w})$$  \hspace{1cm} (5.2)

where $\Phi \hat{w} = R(\Phi \hat{w})$ from equation (3.10). The normalized error induced by truncation of modes is given by

$$\epsilon = \frac{|\dot{r}_f - \dot{\bar{r}}|}{|\dot{\bar{r}}|}$$  \hspace{1cm} (5.3)

The basis functions that are used in the matrix $\Phi$ are the normalized retained modes only. Substituting equations (5.1) and (5.2) into equation (5.3)

$$\epsilon = \frac{|\Phi \Phi^T R (\Phi \hat{w}) - R (\Phi \hat{w})|}{|R (\Phi \hat{w})|}$$  \hspace{1cm} (5.4)

which can be simplified to

$$\epsilon = \frac{|(\Phi \Phi^T - I) R (\Phi \hat{w})|}{|R (\Phi \hat{w})|}$$  \hspace{1cm} (5.5)

This equation can be used to check the normalized error at discrete time instants during the numerical integration and will be referred to as the error norm. The resulting error from equation (5.5) is representative of the potential error that could be introduced by the ROM at each time step. The normalized error only provides an indication of the potential error induced between iterations due to the neglected modes. In essence, this is a comparison between the truncated model and a model that would retain all the reduced order modes.

The individual normalized modal contribution associated with the individual retained mode is given by

$$\kappa_i = \frac{|(\bar{\phi}_i \bar{\phi}_i^T) R (\Phi \hat{w})|}{|R (\Phi \hat{w})|}$$  \hspace{1cm} (5.6)

This number will generally be largest for the first few modes. These first few modes typically have the most information content so should make the largest contribution [2, 8]. This calculation could be used to determine the effectiveness or contribution of individual modes.
5.2 Error Interpretation

In the numerical simulation, the conserved flow variables \( \rho, \rho u, \rho v, \) and \( e \) of the conservative state vector \( q = [\rho \rho u \rho v e] \) are found through numerical integration. The amount of variation in the individual variables across both space and time are quite different. The spatiotemporal variation is usually lowest for density \( \rho \) and highest for conserved energy \( e \). If the variation in the energy is adequately captured by the POD/ROM, there is a reasonable amount of confidence in saying that the POD/ROM will adequately capture the variation in the remaining flow variables [7].

The effectiveness of the reduced order model and the modes that comprise the model can be determined from the calculations of the error norms, modal contributions and residual values. These observations could be used to adjust the model as needed. These adjustments could either be done manually or automatically. The POD/ROM method is called an adaptive POD/ROM if the adjustments are made automatically based on these in-situ error calculations. Adaptive POD is where the model can be changed in the middle of a simulation to better match operating conditions.

The effectiveness of the included modes is determined by the error norm using equation (5.5). The modal contribution of the individual included modes is found using equation (5.6). The individual modal contributions can be used to determine if that particular mode can be removed from the model if its contribution is below a certain tolerance. This tolerance could be derived from observation of the remaining ROM modes. Monitoring residual values given by equation (3.11) could be used to determine if a truncated mode should be included in the ROM. The error norm is then used to determine if truncating a mode or including a mode results in a more effective ROM. A predetermined tolerance level is used to determine when a mode should be included or truncated from the ROM.

It should be kept in mind that these calculated values may vary in time when analyzing the behavior of the modes. In some situations, the error norms, modal contributions, and residuals may contain large oscillations showing that there are times when a particular mode may become more or less significant. Higher modes tend to contain start-up transient data or information on infrequently occurring structures. These modes may become less significant as the solution progresses.
5.3 Boundary Conditions and Experiment Parameters

The difference between the full and reduced order numerical simulations lies in the choice of parameters and the solution method. Boundary conditions are parameters needed to solve the differential equations and affect the outcome of the solution. Altering the boundary conditions changes the outcome of the solution. The numerical simulation must be rerun when boundary conditions or other parameters are changed. Naturally, a wide variety of boundary conditions and design parameters are used when a component subjected to fluid flow is being analyzed. It would be desirable to test as many of these parameter variations as possible using the reduced order simulation to save computational expense.

Two simulations are generally run using the on-design parameters and boundary conditions when a ROM generated by POD is used. The first is done for the generation of the snapshots used to construct the reduced order model. The second simulation that may be performed is used for validation and establishing a baseline for comparing subsequent ROM performances. Accuracy of the reduced order simulation is defined by results very similar to the full order simulation. The similarity is controlled by the amount of retained basis functions or vectors.

The initial state of the flow is defined to be at freestream conditions with the panel undeflected (flat). The freestream conditions are defined by the far-field density $\rho_\infty$, far-field velocity $U_\infty$, and far-field pressure $\rho_\infty U_\infty^2$. These variables are used to non-dimensionalize the flow variables. The flow field values and the non-dimensional far-field values are then

$$
\rho \rightarrow 1, \quad u \rightarrow 1, \quad v \rightarrow 0, \quad p \rightarrow 1/(\gamma M_\infty^2),
$$

where $p$ is the pressure, $(u, v)$ are velocity components in the $(x, y)$ coordinate directions, respectively, $\gamma$ is the ratio of specific heats and $M_\infty$ is the freestream Mach number.

A transpiration boundary condition is used at the bump surface to simulate the bump motion. This condition is applied at the $y = 0$ location. Since the transpiration boundary condition allows the grid to remain stationary in time, the parameters governing the motion of the bump can be changed without disturbing the spatial dependence of POD. The transpiration boundary condition
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involves the enforcement of the exact condition of impermeability at the panel surface at \( y = 0 \),

\[
-u \frac{\partial y_s}{\partial x} + v = \frac{\partial y_s}{\partial t} \quad (y_s = y_s(x, t)),
\]  

(5.8)

Equation (5.8) assumes regularity of the computed solution and smallness of the deformation: \( y_s(x, t) \ll 1 \). Conditions are applied to close the discretized Euler equations at the panel surface in addition to the impermeability condition:

\[
\frac{\partial u}{\partial y} = \frac{\partial p}{\partial y} = \frac{\partial \rho}{\partial y} = 0 \quad (y = 0)
\]  

(5.9)

Derivatives of primitive variables with respect to the \( y \)-coordinate in equation (5.9) are specified to vanish rather than to the coordinate normal to the deformed panel. This approximation assumes smallness of deformation slopes, which is consistent with the prior assumption of \( y_s(x, t) \ll 1 \).

### 5.3.1 On-Design Parameters

The flow above the panel is transonic with a mach number \( M_\infty = 1.2 \) and a ratio of specific heats \( \gamma = 1.4 \). The time step size is \( \Delta t = 0.01 \). The maximum bump amplitude is set to \( \delta_1 = 0.005 \). The amplitude of the bump height is normalized by the chord length (which was defined to be unity). The frequency of the cosine function is \( \omega = 1.0 \) and a modulation parameter of \( \beta = 3 \). These conditions remained unchanged for the validating reduced order simulation. The variable used to identify the reduced order time step is \( \Delta t_r \) to distinguish it from the full order time step. The size of the reduced order time step is \( \Delta t_r = 0.01 \), which remains unchanged from the full order time step size.
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5.3.2 Off-Design Parameters

The parameters $\delta_1$, $\omega$ and $\Delta t_r$ are changed one at a time to test their influence on the accuracy of the solution. The maximum bump amplitude is raised from $\delta_1 = 0.005$ to $\delta_1 = 0.01$. This test matches the test conditions used by Slater [2] and is done to calculate the modal contributions and normalized error to compare to their residual calculations. Other values are tried to test the limits of variation of this variable. The largest amplitude used is $\delta_1 = 0.05$.

The bump frequency $\omega$ is varied from its on-design condition. The values tried are $\omega = 0.5$ and $\omega = 4.0$ which are $\frac{1}{2}$ and $2x$ the on-design values, respectively. This value is expected to have a large influence on the reduced order simulation since the bump function is a non-linear function.

The reason for the use of the POD/ROM method is to increase the time step size in the reduced order simulation. The reduced order time step is increased until the ROM cannot reach a stable conclusion. ROMs with differing numbers of modes are tried to determine if this can restore or reduce stability.
Results From Full and Reduced Order Simulations Using On-Design Conditions

The initial simulations were performed to gather snapshots and verify the functionality of the reduced order model (ROM). These on-design simulations were conducted for 2000 iterations. The first simulation was full order and was performed to gather snapshots and create the basis functions. Snapshots were taken at a period of 25 time steps. This interval was the same interval used by Slater et al. [2]. This rendered 80 snapshots. K-L Analysis was used to render 80 basis vectors (modes). These were then truncated to the first 24 as ranked by corresponding eigenvalue. The first 4, 8, 12 and 16 modes of these 24 were used in the ROMs and the balance of the 24 modes were used in error calculations.

Subsequent reduced order simulations were performed using the 4, 8, 12 and 16 mode ROMs. The results from the reduced order solutions were compared to the full order solution to draw conclusions on the accuracy of the models and effectiveness of the error analysis methods. Similarity between the full and reduced order solutions will be dictated by the number of retained modes in the ROM.

6.1 Results, on-design simulations

Contour plots of the solution provide the quickest visual indicator of the accuracy at the conclusion of the reduced order simulations. These contour plots show the numeric values of the chosen
6.1. RESULTS, ON-DESIGN SIMULATIONS

variable across the entire flow field. The variable chosen for these plots was the pressure, $p$. Pressure was chosen since its value is dependent on the value of all of the conserved flow variables so that any inaccuracy in the solution is magnified by calculating the pressure. The value of the pressure variable from the reduced order solution was compared to the full order solution. The contour locations and values of the reduced order solution were compared to the full order solution to determine if the structures of the flow were captured accurately and to evaluate similarity. Figure (6.1) shows the pressure profile of the full order simulation at on-design conditions. This plot was rendered from the results at the conclusion of the 2000 iteration flow solution over the bump. The pressure profiles from the reduced order simulations are shown in figures (6.2a-6.2d). These figures show the resulting pressure profiles at the end of 2000 iteration reduced order flow simulations. The flow variables for the on-design full and reduced order simulations from section (5.3.1) were $M_\infty = 1.2$, $\gamma = 1.4$, $\Delta t = 0.01$, $\delta_1 = 0.005$, $\omega = 1.0$ and $\beta = 3$.

Figure 6.1: Pressure after 2000 iteration simulation, full order at on-design conditions.
Comparison figures (6.2a-6.2d) to figure (6.1) shows that using the 12 or 16 mode ROM approximates the full order solution better than the 4 or 8 mode ROMs. The solution from the 4 mode ROM shown in figure (6.2a) indicates a relatively large difference between the full and reduced order solution. It does capture some of the features occurring near the surface. However, it exaggerates features away from the bump surface oscillation. The 8 mode ROM shown in figure (6.2b) is lacking some detail downstream of the oscillating bump but does a much better job than the 4 mode ROM. The 12 and 16 mode ROMs perform similarly. They capture the pressure profile accurately based on the contour locations and values.
The normalized error for the 4, 8, 12, and 16 mode simulations is shown in figure (6.3). The normalized error for all the reduced order simulations remains less than one percent. When comparing the pressure plots for the full order and 4 mode reduced order solutions, it appears that the error should be higher. As mentioned in Section 5.1, the normalized error is unrelated to the absolute error. The normalized error from the 4 mode model remains significantly higher than the 8, 12 and 16 mode models. The normalized error from the 8 mode model continues to increase as the number of iterations increases. The 12 and 16 mode ROM’s have nearly identical normalized errors and remain relatively low after the start-up transient.

There appears to be a difference in period of the 8, 2, and 16 mode errors. Performing a spectral analysis on the data vectors used to plot the normalized errors reveals to peaks at $\omega \approx 1.0$ and $\omega \approx 2.0$. A peak at $\omega \approx 1.0$ is the only one present for the 12 and 16 mode normalized error.

The increasing trend in error with the 8 mode ROM in figure (6.3) was studied further. The simulation was extended to 10000 iterations to determine if this trend would continue. The normalized error for the 8 mode ROM for 10000 iterations is shown in figure (6.4)
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The error increases approximately one order of magnitude when the number of iterations is extended. The rate at which the error is increasing is slowing as the number of iteration increase. The pressure at the simulation conclusion is compared to gauge the impact on the solution.

Figure 6.4: Normalized error, 10000 iteration simulation, 8 mode ROM, on-design conditions

Figure 6.5: Pressure, 10000 iteration simulation, full order and 8 mode ROM, on-design conditions

Figures (6.5a-6.5b) shows the resulting pressure plots after 10000 iterations using the full order model and the 8 mode ROM at on-design conditions. The 8 mode ROM does a very poor job of reproducing the pressure profile after 10000 iterations. The difference between the highest contour
values in figures (6.5a) and (6.5b) is less than 10% but greater than 35% for the lowest contour values. Contour locations are also largely different showing nearly no similarity at any location in the flow field.

All the simulations showed a periodic oscillatory behavior. Inclusion of more modes appeared to reduce the oscillations in the normalized error. The 4 mode simulation had the highest sustained normalized error of the reduced order simulations. The 8 mode simulation resulted in lower normalized error. However, the normalized error crept upwards as the solution progressed in time. This may indicate that the 4 added modes contained information pertinent to the start-up transient but lacked in information content relating to the steady-state. The 12 mode ROM decreased the steady-state normalized error and oscillations over the 8 mode simulation. The behavior of the 16 mode ROM followed the same trend as the 12 mode ROM.

The normalized contribution of the individual retained modes to the solution at each time step was calculated using equation (5.6). This provides an indication of the potential significance of each mode to the solution at a particular time in the integration process. Figure (6.6) shows the modal contributions of the 4 modes used in the 4 mode reduced order simulation.

![Figure 6.6: Normalized modal contributions, 4 mode simulation using on-design condition.](image)

The contributions of each of the modes can be seen to have a strongly periodic nature. The modes appear to trade significance to the solution during time integration. Modes 1 and 2 in figure (6.7a) illustrate this point. Although they have the same period, they are shifted such that one peaks when the other is at a valley.
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Figure 6.7: Normalized modal contributions, 8 mode 2000 iteration simulation using on-design condition.

Figures (6.7a-6.7b) show the normalized individual modal contributions from the 8 mode 2000 iteration simulation. The contributions of the individual modes in the 8 mode simulation display the same periodic behavior as the modes in the 4 mode simulation. However, the contribution of the first few modes is slightly lower than in the 4 mode simulation. This is most likely due to contributions of the higher modes. Figures (6.8a-6.8b) show the modal contributions for the 8 mode ROM simulation carried out for 10000 iterations. The contributions of the first 2 modes decrease but the contributions of the remaining modes increase or remain constant as the solution progresses.

Figure 6.8: Normalized modal contributions, 8 mode simulation, 10000 iterations, on-design condition.
Figure 6.9: Normalized modal contributions, 12 mode simulation using on-design condition.

Figures (6.9a-6.9c) show the normalized individual modal contributions from the 12 mode simulation. Modal contribution decreases as the mode number increases. The contribution of the first 8 modes remain consistent throughout the solution. The contribution of the last 4 modes make their most significant contributions at the beginning of the solution and fall off as the solution progresses. This shows that they contain information pertinent to the start-up transient.
Figure 6.10: Normalized modal contributions, 16 mode simulation using on-design condition.

Figures (6.10a-6.10d) show the normalized individual modal contributions from the 16 mode simulation. There is little change in the contributions of the first 12 modes when compared to the contributions of the 12 mode simulation in figures (6.9a-6.9c). The last 4 modes of the 16 mode ROM make their largest contributions at the beginning of the solution showing that they contain more information on the start-up transient and less on the steady state.

The remainder of the 24 modes not used in the reduced order model were tracked by residuals calculated using equation (3.12). The use of residuals provide insight on the potential importance of modes left out of the ROM.
Figures (6.11a-6.11e) show the residual values for the truncated modes for the 4 mode ROM simulation. The residuals for the all the modes increase slightly after the beginning of the solution showing that modes containing information on the steady-state have been left out. Residual values of all the tracked modes remain significant relative to each other and do not show the same decreasing
trend as mode number increases like modal contributions did. For instance, the residual value of mode 9 is higher than the residual values of modes 5 through 8.

Figures (6.12a-6.12d) show the residual values for the truncated modes for the 8 mode ROM 2000 iteration simulation. The residual values are less than the residuals in the 4 mode simulation. Similarly, the residual values of the tracked modes all remain significant when compared to each other. The most important observation of these residuals is their increasing trend as the solution progresses. This indicates that some of these tracked modes are becoming more significant. The simulation was carried out an additional 8000 iterations.
6.1. RESULTS, ON-DESIGN SIMULATIONS

Figure 6.13: Residuals, 8 mode ROM, 10000 iteration simulation at on-design conditions

Figures (6.13a-6.13d) show the residual values for the truncated modes for the 8 mode ROM simulation carried out for 10000 iterations. The residual values have grown very large compared to previous simulations. These residuals are still continuing to increase at the end of the solution.
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**Figure 6.14:** Residuals, 12 mode ROM, 2000 iteration simulation at on-design conditions

Figures (6.14a-6.14c) show the residual values for the truncated modes for the 12 mode ROM simulation. The residuals from this simulation are much lower than those from the 4 and 8 mode ROM simulations. The residuals are highest at the start of the solution. Then they decrease as the solution progresses. Although all the residual values are nearly the same, they are much smaller at the conclusion of the simulation than they were at the start of the simulation.
Figures (6.15a-6.15b) show the residual values for the truncated modes for the 16 mode ROM simulation. The addition of 4 modes to the 12 mode ROM did not significantly affect the residuals in the way which adding 4 modes did to both the 4 and 8 mode ROMs. Little difference can be seen by comparing figures (6.15a-6.15b) to (6.14b-6.14c).

The residuals from the 4 mode simulation are consistently higher than their corresponding residuals in the other 2000 iteration ROM simulations featuring more modes. Adding 4 modes for an 8 mode ROM results in correspondingly lower residual values in the short term. Carrying out the 8 mode ROM simulation to 10000 iterations results in residuals grossly larger than the residual values in all other simulations. This could be seen as a possible instability. The addition of 4 modes to the 8 mode ROM stabilizes the residual values. The resulting residual values from the 12 mode ROM are much lower than the corresponding 8 mode ROM residuals and continue to decrease as the solution progresses. Using 4 more modes in the 16 mode ROM results in little change to the residuals seen in the 12 mode ROM simulation. This could be seen as a point of diminishing returns.

The residual values, modal contributions and normalized errors have been given in the preceding paragraphs. The goal is to use these figures to draw conclusions on the accuracy of the reduced order solution. Comparison of the reduced order pressure plots shown in figures (6.2a-6.2d) to the full order pressure plot in figure (6.1) showed that the 4 mode ROM performed the worst. The 8 mode ROM performed slightly better and the 12 and 16 mode ROM simulations were similar and the best approximation to the full order solution. The poor performance of the 4 mode ROM was
6.1. RESULTS, ON-DESIGN SIMULATIONS

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reflected in a normalized error which was higher than that of the other ROM simulations. It also had consistently high modal contributions and high residual values. The instability in the 8 mode ROM simulation was not readily apparent from the pressure plot after 2000 iterations. The increasing trend in the normalized error and residual values showed that there was an instability associated with this ROM. In both the 12 and 16 mode ROM simulations, behavior of the normalized error, modal contributions and residual values was a decreasing trend as time progressed.

Since the 4 mode reduced order simulations did a poor job of accurately reproducing the full order fluid flow in the on-design testing, the 4 mode ROM was not used in any of the off-design testing. If this ROM could not adequately reproduce the flow from which it was created, it could not be expected to accurately reproduce off-design conditions.
Results From Reduced Order Simulations Using Off-Design Conditions

Various parameters associated with the behavior of the oscillating bump are changed. Simulations under these conditions are called off-design. Reduced order simulations at off-design conditions are run using ROMs generated from an on-design simulation. This is done to generate data for calculating normalized error, modal contributions and residual values. These will then be compared to the data from Chapter 6. The goal is twofold. First, the effect of the change to the variable is compared to the change in the results. For example, what is the change to the normalized error when compared to the on-design normalized error if an on-design parameter is doubled. The second goal is to observe the behavior of the normalized error, modal contributions and residual values; is this behavior (i.e. trending up, trending down) similar for models with the same number of modes but with differing design parameters.

7.1 Results From Variation of Maximum Bump Amplitude, $\delta_1$

The first design parameter to be changed was the maximum bump amplitude $\delta_1$. The on-design maximum bump amplitude was $\delta_1 = 0.005$. The remaining parameters were maintained at the on
7.1. RESULTS, VARIATION OF $\delta_1$

Design conditions. These conditions were $M_\infty = 1.2$, $\gamma = 1.4$, $\Delta t = 0.01$, $\omega = 1.0$ and $\beta = 3$. Larger bump amplitudes were tried. The larger off-design bump amplitudes were $\delta_1 = 0.01$ and $\delta_1 = 0.05$.

![Diagram](image)

Figure 7.1: Pressure after 2000 iteration simulation with $\delta_1 = 0.01$, all other conditions same as on-design.

Figure (7.1a) shows the pressure contour plot of the flow field at the conclusion of the full order 2000 iteration simulation. Figures (7.1b-7.1d) show the pressure contours of the flow field at the conclusion of 2000 iteration simulations using 8, 12 and 16 mode ROMs with a bump amplitude of $\delta_1 = 0.01$. The 4 mode ROM was not used due to poor performance in reproducing the full-order results using the on-design conditions in the previous chapter. Comparing figure (6.1) to figure (7.1a) shows that the increase in the maximum bump amplitude has increased the range in pressure and there is a larger pressure fluctuation downstream but the overall behavior is still the same. The performance of the 8 mode ROM simulation is nearly identical to that of the on-design simulation.
7.1. RESULTS, VARIATION OF $\delta_1$

is figure (6.2b) and does not capture the downstream fluctuation. The 12 and 16 mode ROMs give better performances. Both miss the downstream fluctuations and some details above and away from the panel surface.

![Graph showing normalized error](image)

Figure 7.2: Normalized error during 2000 iteration simulation with $\delta_1 = 0.01$, all other conditions on-design.

The normalized error was monitored during the solution process. Figure (7.2) shows the normalized error for the 8, 12 and 16 mode simulations with the maximum bump amplitude $\delta_1 = 0.01$. Doubling the bump amplitude $\delta_1$ from the on-design condition has had the effect of doubling the normalized error. This is due to the linear terms of the governing equations being dominant [2]. The trend of the 8 mode ROM normalized error is increasing$^1$.

$^1$Increasing the number of iterations of the 8 mode simulation caused an instability that resulted in a crash of the flow solver for any of the off-design conditions. No results are presented for simulations beyond 2000 iterations with the 8 mode ROM and off-design conditions due to this instability.
Figure 7.3: Normalized modal contributions, 8 mode simulation, $\delta_1 = 0.01$, all other conditions on-design.

Figures (7.3a-7.3b) show the modal contributions for the 8 mode ROM 2000 iteration simulation with maximum bump amplitude $\delta_1 = 0.01$. The change to bump amplitude has had little effect on the contributions of the modes when compared to the normalized contributions in figures (6.7a-6.7b).
Figure 7.4: Normalized modal contributions, 12 mode simulation, $\delta_1 = 0.01$, all other conditions on-design.

Figures (7.4a-7.4c) show the modal contributions of the 12 mode ROM 2000 iteration simulation. The increase in bump amplitude has had little effect on the contributions of the modes when compared to the normalized contributions in figures (6.9a-6.9c).
Figure 7.5: Normalized modal contributions, 16 mode simulation, $\delta_1 = 0.01$, all other conditions on-design.

Figures (7.5a-7.5d) show the modal contributions of the 16 mode ROM 2000 iteration simulation. The increase in bump amplitude has had little effect on the contributions of the modes when compared to the normalized contributions in figures (6.10a-6.10d).
7.1. RESULTS, VARIATION OF $\delta_1$

Figures (7.6a-7.6d) show the residual values for the truncated modes for the 8 mode ROM 2000 iteration simulation. The residual values are doubled from those from the on-design conditions shown in figures (6.12a-6.12d).

Figure 7.6: Residuals, 8 mode ROM, $\delta_1 = 0.01$, all other conditions on-design.
7.1. RESULTS, VARIATION OF $\delta_1$

Figure 7.7: Residuals, 12 mode ROM, $\delta_1 = 0.01$, all other conditions on-design.

Figures (7.7a-7.7c) show the residual values for the truncated modes for the 12 mode ROM 2000 iteration simulation. The residual values are doubled from those from the on-design conditions shown in figures (6.14a-6.14c).
7.1. RESULTS, VARIATION OF $\delta_1$

Figure 7.8: Residuals, 16 mode ROM, $\delta_1 = 0.01$, all other conditions on-design.

Figures (7.8a-7.8b) show the residual values for the truncated modes for the 16 mode ROM 2000 iteration simulation. As with the 8 and 12 mode ROMs, the residual values are doubled from those from the on-design condition shown in figures (6.15a-6.15b).

Doubling the size of the bump amplitude from the on-design value serves to double the amplitude of the residuals for all 3 ROMs. The behavior of the residuals remained unaffected meaning that the residuals for the on- and off-design ROM simulations would be nearly identical if the scaling were removed. This doubling corresponds to the doubling of the normalized error and for the same reason as cited in ref. [2]. This linear relation is put to the test by increasing the maximum bump amplitude $\delta_1$ by an order of magnitude.
7.1. RESULTS, VARIATION OF $\delta_1$

Figure 7.9: Pressure after 2000 iteration simulations with $\delta_1 = 0.05$, all other conditions same as on-design

Figure (7.9a) shows the pressure contour of the flow field at the conclusion of the 2000 iteration full order simulation. The larger maximum amplitude has increased the range in pressure values over those in the on-design simulation in figure (6.1). Figures (7.9b-7.9d) show the pressure contours of the flow field above the panel after reduced order 2000 iteration simulations using 8, 12 and 16 modes with a maximum bump amplitude of $\delta_1 = 0.05$. None of these ROMs accurately capture the pressure profile of the flow. They all underestimate the maximum pressure and overestimate the minimum pressure. The general trends near the bump surface are adequately captured but inaccuracy occurs away from the bump surface.
7.1. RESULTS, VARIATION OF $\delta_1$

The effect of raising the maximum bump amplitude $\delta_1$ by an order of magnitude has had almost the same effect as doubling it from the on-design condition. The change has been nearly linear in that the normalized error has been increased by an order of magnitude from figure (6.3). Differences have developed in the form of oscillations of greater amplitude in the ROM simulations and a slightly upward trend in the 12 and 16 mode ROM simulations.
7.1. RESULTS, VARIATION OF $\delta_1$

Figure 7.11: Normalized modal contributions, 8 mode ROM, 2000 iteration simulation, $\delta_1 = 0.05$, all other condition on-design.

Figures (7.11a-7.11b) show the modal contributions during the 8 mode simulation for the maximum bump amplitude $\delta_1 = 0.05$. The change to bump amplitude has had little effect on the contributions when compared to figures (6.7a-6.7b).
7.1. RESULTS, VARIATION OF $\delta_1$

Figure 7.12: Normalized modal contributions, 12 mode simulation, $\delta_1 = 0.05$, all other condition on-design.

Figures (7.12a-7.12c) show the modal contributions of the 12 mode simulation for the maximum bump amplitude $\delta_1 = 0.05$. Comparison of these figures to the on-design contributions in figures (6.9a-6.9c) shows the most difference in the contributions of the higher modes. Modes 9 through 11 start to regain significance after their initial drop-off after the start-up transient, whereas they remained small after the start-up transient in the on-design simulation.
7.1. RESULTS, VARIATION OF $\delta_1$

Figures (7.13a-7.13d) show the modal contributions of the 16 mode simulation for the maximum bump amplitude $\delta_1 = 0.05$. Comparison of these figures to the on-design contributions in figures (6.10a-6.10d) show the most difference in the contributions of the higher modes. The normalized contributions of modes 9 through 11 grow after their initial drop-off after the start-up transient. Mode 13 remains much more significant than it was in the on-design simulation. The contributions of the remaining higher modes were near their on-design counterparts.
7.1. RESULTS, VARIATION OF $\delta_1$

Figure 7.14: Residuals, 8 mode ROM, 2000 iteration simulation, $\delta_1 = 0.05$, all other conditions on-design.

Figures (7.14a-7.14d) show the residual values for the truncated modes for the 8 mode ROM simulation with $\delta_1 = 0.05$. Increasing the maximum bump amplitude by a factor of ten has increased the residual values by a factor of ten when compared to figures (6.12a-6.12d), the on-design residual values. Differences in the trends of some of the residuals also appear. These differences are not as subtle as the case with the normalized error.
Figures (7.15a-7.15c) show the residual values for the truncated modes for the 12 mode ROM simulation with $\delta_1 = 0.05$. Multiplying the maximum bump amplitude by ten from the on-design condition shows a corresponding increase in the residuals by a factor of ten. The trend of the residuals in fig. (7.15a) are similar to the residuals trend in fig. (6.14a). Differences in trends appear when comparing figures (7.15b) and (7.15c) to (6.14b) and (6.14c). The residual values of on-design simulation oscillate about a zero mean whereas the off-design simulation with the larger amplitude $\delta_1$ display a trend diverging from zero.
7.1. RESULTS, VARIATION OF $\delta_1$

![Graph](image)

(a) Modes 17-20  
(b) Modes 21-24

Figure 7.16: Residuals, 16 mode ROM, 2000 iteration simulation, $\delta_1 = 0.05$, all other conditions on-design.

Figures (7.16a-7.16b) show the residual values for the truncated modes for the 16 mode ROM simulation with $\delta_1 = 0.05$. Comparing figures (7.14a-7.16b) to the residuals from the on-design simulation in figures (6.12a-6.15b) show some differences. There has been a corresponding linear increase in the residual values from the on-design values. The residuals in this off-design simulation did decrease sharply after the start-up transient but oscillated more in the steady-state. Modes 18, 19, 22 and 24 from the off-design simulation diverge from the zero mean that they had in the on-design simulation.

Increasing the maximum bump amplitude $\delta_1$ by an order of magnitude had nearly the same effect as doubling it from the on-design value. The normalized error and residual values showed a linear increase corresponding to the change in $\delta_1$. The modal contributions remained nearly the same for the first 8 modes of the ROMs and showed some increases in the higher modal contributions. Observation of the pressure contour plots showed that the ROM solutions were not achieving accurate solutions for this value of $\delta_1$, but nothing is immediately obvious from observation of the normalized error, modal contributions or residual values. The increasing trend of the residual values and normalized contributions of the higher modes may provide a clue that the solution is headed towards inaccuracy.
7.2 Results from Variation of the Bump Frequency, $\omega$

The parameter governing the bump frequency $\omega$ was varied from its on-design value. The on-design bump frequency was $\omega = 1.0$. Values above and below the on-design condition were tried. The outer limits of accuracy were around $\omega = 0.5$ and $\omega = 2.0$. These values were one-half and twice the on-design value, respectively. All other parameters were kept at the on-design conditions.

![Pressure profiles](a) Full order simulation
![Pressure profiles](b) 8 mode simulation
![Pressure profiles](c) 12 mode simulation
![Pressure profiles](d) 16 mode simulation

Figure 7.17: Pressure after 2000 iteration full and reduced order simulations, bump frequency $\omega = 0.5$, all other parameters on-design

Figures (7.17a-7.17d) show the pressure profile over the panel for the full order and 8, 12 and 16 mode reduced order simulations. The ROMs accurately reproduce the pressure profile over the bump. All three estimate a pressure increase downstream of the bump along the panel surface when the full order solution shows a pressure drop. The pressure contours away from the panel surface are not accurately represented by any of the ROMs.
Figure 7.18: Normalized error, 2000 iteration simulations with 8, 12 and 16 mode ROMs, $\omega = 0.5$, all other conditions on-design

Figure (7.18) shows the normalized error for the 8, 12, and 16 mode reduced simulations with $\omega = 0.5$. Comparison of the normalized errors from the on-design simulation in figure (6.3) shows that decreasing the bump frequency has affected the normalized error. The period of the oscillations in the errors has increased and the error norms are larger. The increase for the 8 mode simulation was not that significant as most of its increase was in the first part of the simulation and was about twice that of the on-design simulation at the end. The increase for the 12 and 16 mode simulations was almost an order of magnitude throughout the simulation but does not appear to be increasing.
7.2. RESULTS, VARIATION OF $\omega$

Figure 7.19: Normalized modal contributions, 8 mode simulation, $\omega = 0.5$.

Figures (7.19a-7.19b) show the modal contributions of the 8 mode simulation with $\omega = 0.5$. Comparison of these contributions the on-design modal contribution in figures (6.7a-6.7b) shows that corresponding modes make the same levels of contribution. The effect of decreasing the bump frequency is an increase in the period of the modal contributions.
7.2. RESULTS, VARIATION OF $\omega$

Figure 7.20: Normalized modal contributions, 12 mode simulation, $\omega = 0.5$.

Figures (7.20a-7.20c) show the modal contributions of the 12 mode simulation with $\omega = 0.5$. Contribution levels of the first 4 modes are the same as in the on-design simulation shown in figures (6.9a-6.9c). The subsequent modes have a much higher contribution during the solution process, most notably modes 9 and 11 which remain significant and do not taper off as do their on-design counterparts. The effect of decreasing the bump frequency is an increase in the period of the modal contributions.
7.2. RESULTS, VARIATION OF $\omega$

Figures (7.21a-7.21d) show the modal contributions of the 16 mode simulation with $\omega = 0.5$. The contribution levels of the first 4 modes are the same as in the on-design simulation shown in figures (6.9a-6.9c). Modes 5 through 8 have slightly higher contributions. Modes 9 through 13 remain high and do not taper off like their respective contributions in the on-design simulation. The effect of decreasing the bump frequency is an increase in the period of the modal contributions.

Figure 7.21: Normalized modal contributions, 16 mode simulation, $\omega = 0.5$. 
7.2. RESULTS, VARIATION OF $\omega$

Figure 7.22: Residuals, 8 mode ROM, 20 second simulation, $\omega = 0.5$, all other conditions on-design.

Figures (7.22a-7.22d) show the residual values for the truncated modes for the 8 mode ROM simulation with $\omega = 0.5$. Residual values are higher but less than twice the on-design residuals in figures (6.12a-6.12d). The reduction in bump frequency $\omega$ has added another frequency to the behavior of the residual values.
Figure 7.23: Residuals, 12 mode ROM, 20 second simulation, $\omega = 0.5$, all other conditions on-design.

Figures (7.23a-7.23c) show the residual values for the truncated modes for the 12 mode ROM simulation with $\omega = 0.5$. Decreasing the bump frequency has increased the residual values over those of the on-design simulation shown in figures (6.14a-6.14c). The increase, depending on residual, is about one order of magnitude or less. The residual values do not taper off as they did in the on-design ROM simulation.
Figure 7.24: Residuals, 16 mode ROM, 20 second simulation, $\omega = 0.5$, all other conditions on-design.

Figures (7.24a-7.24b) show the residual values for the truncated modes for the 16 mode ROM simulation with $\omega = 0.5$. Residuals 17 through 20 remain high and significant throughout the solution when compared to the residuals in figures (6.15a-6.15b). Residual 21 to 23 oscillate about a zero mean and are closer to the on-design ROM residual values. Mode 24 has a residual value that is almost ten times its on-design residual value.
7.2. RESULTS, VARIATION OF $\omega$

Figure 7.25: Pressure after 2000 iteration full and reduced order simulations, bump frequency $\omega = 2.0$, all other parameters on-design

Figures (7.25a-7.25d) show the pressure profile over the panel for the full order and 8, 12 and 16 mode reduced order simulations with $\omega = 2.0$. All three ROMs reasonably approximate the range in pressures. They also approximate the behavior of the flow field at the front and back of the bump at the panel surface. The 8 mode ROM misses the pressure drop at the center of the bump. All three ROMs predict pressure drops downstream of the bump that are not present in the full order simulation. None of the ROMs accurately portray the flow field away from the panel surface.
7.2. RESULTS, VARIATION OF $\omega$

Figure 7.26: Normalized error, 2000 iteration simulations with 8, 12 and 16 mode ROMs, $\omega = 2.0$, all other conditions on-design

Figure (7.26) shows the normalized error for the 8, 12, and 16 mode reduced simulations with $\omega = 2.0$. The error for the 8 mode simulation is higher than that in the on-design simulation shown in figure (6.3) but is still of the same order of magnitude. The normalized error for the 12 and 16 mode ROMS are an order of magnitude larger than their on-design counterparts. The increased frequency has also shown up as an increase in the frequency of the oscillations of the error norms.

Figure 7.27: Normalized modal contributions, 8 mode simulation, $\omega = 2.0$. 

(a) Modes 1-4

(b) Modes 5-8
Figures (7.27a-7.27b) show the modal contributions of the 8 mode simulation with $\omega = 2.0$. Increasing the frequency has a more pronounced effect on the contributions when compared to figures (6.7a-6.7b). Modes 6 and 8 have a larger contribution towards the end of the simulation. The contributions of the first 4 modes taper slightly as the simulation progresses.

![Mode Contributions](image1)

(a) Modes 1-4

![Mode Contributions](image2)

(b) Modes 5-8

![Mode Contributions](image3)

(c) Modes 9-12

Figure 7.28: Normalized modal contributions, 12 mode simulation, $\omega = 2.0$.

Figures (7.28a-7.28c) show the modal contributions of the 12 mode simulation with $\omega = 2.0$. The contribution levels of the first 4 modes are slightly lower than in the on-design simulation shown in figures (6.9a-6.9c). Mode 6 has become more significant than mode 5 and modes 9 through 12 remain significant throughout the solutions whereas these tapered off in the on-design simulation.
7.2. RESULTS, VARIATION OF $\omega$

Figures (7.29a-7.29d) show the modal contributions of the 16 mode simulation with $\omega = 2.0$. Contribution levels of the first 8 modes are the same as in the on-design simulation shown in figures (6.10a-6.10d). Modes 9 through 16 remain significant throughout the solution and do not taper off as do the corresponding contributions in the on-design simulation.
7.2. RESULTS, VARIATION OF $\omega$

Figures (7.30a-7.30d) show the residual values for the truncated modes for the 8 mode ROM simulation with $\omega = 2.0$. Doubling the amplitude appears to almost double the residual values of the on-design simulation shown in figures (6.12a-6.12d). The increasing trend is still intact. The increased frequency has been blended into the behavior of the residuals as was the case with $\omega = 0.5$ in figures (7.22a-7.22d). This is most obvious when comparing figure (6.12c) to (7.30c).

Figure 7.30: Residuals, 8 mode ROM, 20 second simulation, $\omega = 2.0$, all other conditions on-design.
7.2. RESULTS, VARIATION OF $\omega$

Figure 7.31: Residuals, 12 mode ROM, 20 second simulation, $\omega = 2.0$, all other conditions on-design.

Figures (7.31a-7.31c) show the residual values for the truncated modes for the 12 mode ROM simulation with $\omega = 2.0$. Although these residuals oscillate about a zero mean like their on-design counterparts in figures (6.14a-6.14c), they oscillate with much higher amplitude and do not taper off as the solution progresses. Of note is the residual of mode 24 which is higher than most of the other residuals.
7.2. RESULTS, VARIATION OF $\omega$

Figures (7.32a-7.32b) show the residual values for the truncated modes for the 16 mode ROM simulation with $\omega = 2.0$. The increase in the bump frequency $\omega$ has resulted in residual values which remain high throughout the solutions instead of tapering off as they did in the on-design reduced order simulations as shown in figures (6.15a-6.15b). This increase is an order of magnitude or less than the on-design residual values.

Changes to the frequency appear in the oscillatory behavior of the normalized error, modal contributions and residual values. The periods and amplitudes were affected. Periods tended to increase for decreased frequency and decrease for increased frequency. The periodicity from the on-design ROM condition was also present demonstrating that the ROMs have some memory of their on-design roots. Amplitude tended to grow for any change to the frequency. Increased frequency caused a larger change to amplitude. The key to observe here is the sustained higher error norms and residual values when their associated on-design ROM counterparts tapered off to relatively lower levels.

Figure 7.32: Residuals, 16 mode ROM, 20 second simulation, $\omega = 2.0$, all other conditions on-design.
Results from Variation of the Reduced Order Time Step, $\Delta t_r$

The number of iterations used in the off-design simulations was determined by the size of the time step. For instance, if the time step used in the ROM simulation was twice the size of the time step used in the full-order simulation, then only half the iterations are needed to reach the same solution point.

The ability to increase the size of the time step is the principal reason for using the POD method. In the full order simulation, the time step size was not able to be increased past a certain point and still run the code to a stable conclusion. Stability was defined as running without a Not a Number (NaN) or division by zero error. The time step size in the RAPOD code was able to be increased with the aid of the reduced order models. Stable results were achieved using a time step size of $\Delta t_r = 0.05$ with the 8, 12 and 16 mode ROMs. This time step represents an 80\% reduction in simulation time. None of the models were stable at a time step larger than $\Delta t_r = 0.05$. The time step size used in the on-design reduced order simulations was $\Delta t_r = 0.01$. All other design parameters were maintained at the on-design conditions.

$^1$4 mode ROM ran stably up to a $\Delta t_r = 0.1$. Results are presented in the appendix but are not presented here due to inaccuracy.
Figure 8.1: Plot of pressure with $\Delta t_r = 0.05$, all other conditions on-design

Figure (8.1a) shows the pressure contour for the full order simulation. Figures (8.1b-8.1d) show the pressure contours of the flow field for a 400 iteration simulation. The 8 mode simulation is inaccurate. It overestimates the minimum and maximum pressures by greater than 7% and is lacking any detail downstream from the bump. The 12 and 16 mode ROMs perform much better. They accurately capture the ranges in pressure and the detail in the pressure contours.
Figures (8.2a-8.2b) show the normalized error for the 8, 12 and 16 mode ROM simulations. The error from the 8 mode simulation shows that the solution is going unstable. This error was higher than any seen in all the previous simulations. The error norms for the 12 and 16 mode simulations are about an order of magnitude above those from the on-design ROM simulation shown in figure (6.3). However, this error drops off rapidly after the start-up transient and tracks like it did in the on-design simulation.

Figures (8.3a-8.3b) show the modal contributions of the 8 mode ROM 400 iteration simulation. The significance of the first 4 modes tapers off slightly as the solution progresses and the contributions of the last 4 modes increase towards the end of the simulation. These contributions remained
steady in the on-design simulation shown in figures (6.7a-6.7b).

![Mode Contributions](image)

**Figure 8.4:** Normalized modal contributions, 12 mode simulation, $\Delta t_r = 0.05$, all other conditions on-design.

Figures (8.4a-8.4c) show the modal contributions of the 12 mode simulation. Modal contributions are nearly identical to the on-design simulation shown in figures (6.9a-6.9c).
Figure 8.5: Normalized modal contributions, 16 mode simulation, $\Delta t_r = 0.05$, all other conditions on-design

Figures (8.5a-8.5d) show the modal contributions of the 16 mode simulation with $\Delta t_r = 0.05$. Modal contributions are nearly identical to the on-design simulation shown in figures (6.10a-6.10d) like the 12 mode simulation.
Figure 8.6: Plot of residuals, 8 mode simulation, $\Delta t_r = 0.05$, all other conditions on-design.

Figures (8.6a-8.6d) show the residuals for the truncated modes of the 8 mode simulation. These residuals are significantly higher than the on-design residuals shown in figures (6.12a-6.12d). Residual values this high compared to the on-design residual values show that this solution is erroneous.
Figures (8.7a-8.7c) show the residuals for the truncated modes of the 12 mode simulation. Residual values are nearly identical to those from the on-design simulation shown in figures (6.14a-6.14c).
Figures (8.8a-8.8b) show the residuals for the truncated modes of the 16 mode simulation with $\Delta t_r = 0.05$. These residuals are similar to those from the on-design simulation shown in figures (6.15a-6.15b). The residuals of the first 4 modes are nearly identical to the on-design simulation. The last 4 follow the same trend but are a little less than twice the on-design residual values.

The pressure plots show that the 8 mode ROM does not accurately capture the flow and the 12 and 16 mode ROM perform satisfactorily. These observations are confirmed by the normalized error, modal contribution and residual value plots. The plots for the 8 mode ROM simulation are going unstable with values which are increasing as the solution progresses. The plots for the 12 and 16 mode simulations are nearly identical to the corresponding plots formed in the on-design simulation in chapter 6.
Conclusions

Proper Orthogonal Decomposition (POD) was used to obtain a low-dimensional model of a high dimensional system. The governing equations of a fluid dynamics problem were projected onto this model. The number of basis vectors used in the model was varied so that their influence on solution accuracy could be observed. Normalized error, normalized modal contribution and residual values of truncated modes were used to draw conclusions on the accuracy of the solution in-situ. It is possible to determine accuracy of the solution using these calculations as described below.

Simulations were conducted using a variety of off-design conditions with the reduced order models. Changing the individual design parameters showed how sensitive the models were to a particular design parameter and how the size of the model affected the performance of the simulation for a given off-design parameter. Data from these simulations was presented in the form of plots of flow solution (pressure), normalized error, residual value and modal contributions. Observations of the behavior of the solutions by the use of normalized errors, residuals and modal contributions were made with these plots and illustrations. The intent of the work presented was in establishing some form of correlation that contributes to the understanding of mode selection through the use of these values and determining accuracy of the solution.

The normalized error provided a means of determining if the addition of more modes had a positive or negative impact on the solution. This impact was relative to the error norm of a previous reduced order solution using the same design conditions simulated with more or less modes. In most of the simulations it was seen that there was a significant decrease in normalized error going from the 8 mode to the 12 mode ROM. This drop was much smaller going from the 12 to 16 mode ROM. This indicates a point of diminishing returns. Scaling of the error norm was present when the design...
variables were changed from their on-design conditions. It is important to note that this error norm was calculated using the reduced order solution and does not reflect the error between the full and reduced order solutions. It only gives the potential error being induced in the reduced order solution at a particular point in time due to truncation of modes. Solution accuracy may be determined by the use of this value. This value provides a minimally computationally expensive method to monitor solution health without the need to ponder the individual modes residual value/normalized contribution. It can be seen in the off-design simulations that the accuracy was degraded when the normalized error was approximately twice its value from the on-design condition.

Residual values offered insight on ROM restructuring. The residual values for all the 8 mode simulations remained high and increasing as the solution progressed. This points to inaccuracy or potential instability in the model. The residuals were reduced when the model size was increased to 12 and 16 modes. Like the error norms, a point of diminishing returns could be seen by comparing the drop in residual values between the 8, 12 and 16 mode simulation residuals. This drop was higher between the 8 and 12 mode simulations and lower between the 12 and 16 mode simulations. The residual values were scaled by the change to the design variable. Inaccuracy appeared whenever residual values were increased or remained significant when corresponding residuals from on-design simulations remained low or tapered off.

Both the normalized error and residual values for the 12 and 16 mode ROMs increased when the design variables were increased to the point where accuracy was degraded. However, this increase was at most one order of magnitude higher than the error norms and residual values of the on-design verification simulation. This can make using these in a numerical model correction step difficult based on magnitude alone. The normalized error and residuals did share one trait when the accuracy decreased. The amount of oscillation increased as the design variable was varied further from the on-design value. The amount of oscillation could be used as a method of determining both accuracy and success of mode addition. Any increasing trend means inaccuracy, vis a vis the 8 mode ROM simulations.

Modal contributions do not provide the same insight to solution accuracy like the residual values and normalized error. Of note was the fact that increasing trends denoted inaccuracy here as
well. The strength in these calculations is in their ability to determine when the significance of a mode is such that it may be pared from the ROM.

The potential use of the combination of these three methods lies in an adaptive or in-situ POD code. Adaptive POD would be were the POD/ROM is changed during the numeric simulation to better suite the operating condition based on error or residual calculations. One method of use of the three calculations presented in this work is as follows. The residuals would be used to determine if too many modes were truncated. This would be done by setting a residual tolerance. Modes above this tolerance are included in the model. In this work, inclusion of more modes always caused a decrease in residual value. This tolerance could be an absolute value or a percentage based on an averaged value. An adaptive value would be necessary to prevent overcorrecting (consider the plateauing trend of the residual values, where inclusion of more modes does not significantly reduce residual). The error norm would be used to determine if the included modes had a positive effect. If the error decreases, the effect was positive and negative if the error increases. The modal contributions would be used to determine if a retained mode is a candidate for truncation. This would allow modes that were significant during transient periods to be removed once the transient period was over.

In the process of generating data for the residual, normalized error and modal contribution calculations, several different system parameters were tested. It was discovered that the bump amplitude parameter $\delta_1$ could be largely varied but the bump frequency parameter $\omega$ was only capable of small variations. There also appears to be some memory in the ROMs. This showed up in the pressure contour plots as having patterns resembling on-design conditions. The memory appears in the contribution, error and residual plots with oscillating periods the same as in corresponding on-design contribution, error and residual plots. This helps underscore the linearity of the POD method. Future work using an automated or adaptive POD/ROM must determine how to find these limits and not exceed them in their simulations. The use of error norms and residual values provides a means to do this.
Computer Programs

The use of computers were indispensable during the course of this thesis work. The main code used for the simulations was RAPOD, written by Dr. Philip S. Beran and Dr. Chris Pettit. This program is written in the FORTRAN95 programming language. Two different compilers were used to compile this code. Compaq Visual Fortran (CVF), Version 6.6 was used first since this was the program used to write the original code. The code was adapted to run on machines not having CVF by replacing the CMXL library routine with LAPACK and BLAS routines, which are publicly available through GPL. This code was compiled using Lahey95, since there is no readily available and reliable free FORTRAN95 compiler.

MATLAB, version 7.0 was used extensively for data processing of the output from the numeric simulations. Most of the plots contained in this thesis were created using MATLAB. Many of the subroutines used were furnished by Dr. Slater from his work in [2].

TECPlot was used as another visualization tool. Multiple data sets can be imported into this program and compared quickly to determine the differences between simulations. Additionally, it allows movie files to be made of the data, which gives a visual time history of the solution.

\LaTeX{} and WinEdt were used for the generation of this report. Reference [29] has proven an invaluable tool in creating this document.
Spectral Analysis

Spectral analysis provides a method for finding the dominant frequencies inherent in a set of discrete data. The interest in using spectral analysis in this research is for investigating a potential method of determining snapshot capture frequency. As mentioned in Section 3.2, it is desirable to have snapshots that contain a sufficient amount of data to create a subspace that will adequately span the original system (i.e. as linearly independent as possible). This can be done only if each of the snapshots contain some new information about the system. If the sampling period happened to be chosen at a frequency inherent to the system, the snapshots taken may turn out to be less independent and lacking in information content. This would result in a poor model.

The system under consideration is a numerically simulated fluid flow. Some CFD solutions display a periodic nature. These are seen as repeating characteristics or coherent structures. The fluid flow studied in this research has such a condition. The periodic nature of the structures in this solution stem from the periodic oscillating nature of the panel. This periodic motion is naturally transmitted to the fluid flowing above it. Spectral analysis is used to determine the frequencies at which these repetitive characteristics occur and the importance or weighting of these frequencies. The anticipation is that the largest spectral density will occur at the oscillating frequency of the panel. The frequencies rendered from the analysis would not be the desirable sample frequencies, but rather frequencies at which samples should not be consistently taken.

Spectral analysis is applied to the discrete data generated at arbitrary points in the grid as follows. A point on the grid and variable is selected so that the flow variable will have a large enough variation in its values over time to keep numerical error in the spectral analysis calculation to a minimum. Points above and downstream of the oscillating bump are considered in this problem.
The energy flow variable is chosen. The values of the flow variable from the chosen grid point is tracked over time and placed in a column vector. If multiple points are chosen, the resulting matrix of column vectors represents the time history of the selected grid points. A Fast Fourier Transform (FFT) is then applied to the data. The power spectral density (PSD) of the transformed data is then calculated, which returns the frequencies at which the most so-called power exists. This is not actual power but an indication that significant events are occurring in the flow at certain frequencies and at the associated grid point. The one-sided spectral density given by

\[
S_{xx}(j\omega) = \begin{cases} 
G_{xx}(j\omega) & 0 = \omega \\
2G_{xx}(j\omega) & 0 < \omega 
\end{cases} \tag{A.1}
\]

where \(G_{xx}(j\omega)\) is the two-sided PSD defined by

\[
G_{xx}(j\omega) = \frac{1}{T} \bar{X}(j\omega)X(j\omega) \tag{A.2}
\]

\(X(j\omega)\) is the Fourier transformed data, and the overbar denotes the complex conjugate.

A drawback to performing a spectral analysis is that it adds additional computational expense. Currently, this step is done as a post-processing step in MATLAB between the training and reduced order simulations. Although the method is easily scripted, it still requires user interface to initiate the code and process the results. Another shortcoming results from the FFT. Due to the nature of the FFT, frequencies in the flow above one-half the sampling frequency will be misrepresented. However, for the case studied here, not much (if anything) is expected to occur at these frequencies. Results from the spectral analysis can be found in appendix A.

A spectral analysis was performed on the data taken from the on-design simulations. A point on the spacial grid was chosen and the time history of the flow variables at that point was recorded. The frequency content of the signal was then extracted from the data set using a MATLAB code developed by Dr. Joseph Slater, called "asd.m". This algorithm utilizes a Fast Fourier Transform (FFT) to extract the frequency characteristics from the data. Figure (A.1a) shows the one-sided auto spectral density (ASD) for the full-order simulation. Figure (A.1b) is a close up of the of the ASD.
The highest peak occurred at the driving frequency of the flow, \( \omega = 1.0 \), the frequency at which the panel oscillates.

The influence of the reduced order simulations on the spectral density of the solution was checked. Figures (A.2a-A.2b) show the ASD for the 8 mode simulation, figures (A.3a-A.3b) show the ASD for the 12 mode simulation, and figures (A.4a-A.4b) shows the ASD for the 16 mode simulation.

Figure A.1: One-sided ASD of full order simulation, on-design parameters

Figure A.2: One-sided ASD of 8 mode ROM simulation, on-design parameters
Comparison of the peaks of the auto spectral densities of the reduced order models in figures (A.2b, A.3b, A.4b) shows that as more modes are retained, more frequency content near the peak at 1 Hz is filtered out. The height of the peak at 1Hz was not necessarily affected.
4 Mode Reduced Order Simulation,

Larger Time Step $\Delta t_r$

Although the 4 mode simulation did not yield the best results, the pressure profile rendered is shown in figure (B.1).

The 4 mode ROM over-predicts the range in pressure by about 4% at the high end and about 2% at the low end when compared to the full order results in figure (6.1). Pressure contour trends near the bump surface are accurately captured. The model also captures the characteristics of the
flow downstream of the bump along the panel surface but is inaccurate above the panel. If the interest is in near panel behavior, this simulation may be adequate.

Figure B.2: 20 second 4 mode ROM simulation, $\Delta t_r = 0.1$, all other conditions on-design

The normalized error is shown in figure (B). The error is an order of magnitude higher than the on-design ROM normalized error shown in figure (6.3). The error follows the same trend but with slightly higher oscillation amplitude.
Bibliography


