Exponential Distributions

Exponential Distribution -

Deals with the distance or time between successive occurrences within an interval of a Poisson Process.

Examples

Time between phone calls
Time between cars arriving at a toll both per hour
Time between failures of electronic components
Distance between successive bumps along a highway
Number of square yards of fabric per flaw
Number of cubic meters of sea water per piece of debris

Conditions

Given a Poisson Process with mean λ, then the random variable X,
(which equals the distance or time between successive occurrences within a Poisson Process),
has an Exponential Distribution with parameter λ such that;

Probability Density Function $f(x) = \lambda e^{-\lambda x}$ for $0 \le x < \infty$

Mean: $\mu = E(X) = 1/\lambda$ Variance: $\sigma^2 = V(X) = 1/\lambda^2$

$$P(a \le X \le b) = \int_a^b \lambda e^{-\lambda x}\,dx = e^{-\lambda a} - e^{-\lambda b}$$

$$P(X \le x) = \int_0^x \lambda e^{-\lambda x}\,dx = 1 - e^{-\lambda x}$$

$$P(X \ge x) = \int_x^\infty \lambda e^{-\lambda x}\,dx = e^{-\lambda x}$$

Solving problems by the exponential approach

Probability of no events in interval x, use $P(X > x) = e^{-\lambda x}$.

Probability of at least one event in interval x, use $P(X < x) = 1 - e^{-\lambda x}$.

Probability of the first event in interval x, use $P(X < x) = 1 - e^{-\lambda x}$.

Probability of operating at least x hours (or more than x hours or no less than x hours), use $P(X > x) = e^{-\lambda x}$.

Probability of operating less than x hours (or no more than x hours or at most x hours), use $P(X < x) = 1 - e^{-\lambda x}$.

Probability of failing within x hours (or failing in less than x hours or failing in the next x hours) is the same as operating for less than x hours, use $P(X < x) = 1 - e^{-\lambda x}$.

ability of failing after x hours (or failing in more than x hours or of not failing within the next x hours) is the same as operating for more than x hours, use $P(X > x) = e^{-\lambda x}$.

Note:
Problems dealing with two or more events in a time interval, need to be worked using the Poisson Distribution probability formula.

Hints for Solving Poisson Processes and Exponential Distribution Problems

1. Suppose the number of phone calls arriving at switchboard follow a Poisson Process with an average of 3 calls per 60 minutes (λ = 3 calls per 60 minutes).

a. What is the probability that there are no calls within a 15 minute period?

Poisson Approach

λ = 3 calls per 60 minutes
λ' / λ = New Interval / Old Interval $\lambda' = \lambda$ (New Interval / Old Interval)
λ' = 3 calls (15 minutes / 60 minutes) = 0.75 calls per 15 minutes

Probability of no calls = P(X = 0)

$$P(X = x) = \frac{e^{-\lambda}\lambda^{X}}{x!} \qquad P(X = 0) = \frac{e^{-0.75}(0.75)^{0}}{0!} = e^{-0.75} = 0.4724$$

Exponential Approach

Although the problem is stated in terms of a Poisson Process (calls per hour), we can also address the question as a time interval problem using the exponential distribution.

"What is the probability of no calls within 15 minutes" is same as asking "what is the probability that the next phone call will not occur until at least 15 minutes has elapsed";
that is to ask - what is P(X > 15 minutes) ?

$$P(X > x) = e^{-\lambda x} \qquad P(X > 15) = e^{-(3/60)15} = e^{-0.75} = 0.4724$$

b. What is the probability that at least one call arrives within a 25 minute period?

Poisson Approach

λ = 3 calls per 60 minutes
λ' / λ = New Interval / Old Interval $\lambda' = \lambda$ (New Interval / Old Interval)
λ' = 3 calls (25 minutes / 60 minutes) = 1.25 calls per 25 minutes
Probability of at least one = $P(X \geq 1) = 1 - P(X < 1) = 1 - P(X = 0)$

$$1 - P(X = 0) = 1 - \frac{e^{-\lambda}\lambda^{0}}{0!} = 1 - \frac{e^{-1.25}\lambda^{0}}{0!} = 1 - e^{-1.25} = 1 - 0.2865 = 0.7135$$

Exponential Approach

"What is probability of at least one call within 25 minutes: is the same as asking "what is the probability of a call arriving in less than 25 minutes"; that is to ask, what is P(X < 25 minutes) ?

$$P(X < x) = 1 - e^{-\lambda x} = 1 - e^{-(3/60)25} = 1 - e^{-1.25} = 1 - 0.2865 = 0.7135$$

Additional Hints for Solving Poisson and Exponential Distribution Problems

Poisson Processes deal with the number of events per interval.

Exponential Distributions deal with the length of the interval between events.

Intervals can be any dimension: time, length, area, volume. The following examples use time.

Solving problems by the exponential approach

Probability of no events in interval x, use $P(X > x) = e^{-\lambda x}$.

Probability of at least one event in interval x, use $P(X < x) = 1 - e^{-\lambda x}$.

Probability of the first event in interval x, use $P(X < x) = 1 - e^{-\lambda x}$.

Probability of operating at least x hours (or more than x hours or no less than x hours), use $P(X > x) = e^{-\lambda x}$.

Probability of operating less than x hours (or no more than x hours or at most x hours), use $P(X < x) = 1 - e^{-\lambda x}$.

Probability of failing within x hours (or failing in less than x hours or failing in the next x hours) is the same as operating for less than x hours, use $P(X < x) = 1 - e^{-\lambda x}$.

Probability of failing after x hours (or failing in more than x hours or of not failing within the next x hours) is the same as operating for more than x hours, use $P(X > x) = e^{-\lambda x}$.

Note:
Problems dealing with two or more events in x hours, need to be worked using the Poisson Distribution probability formula.

Even More Additional Hints for Solving Exponential Distribution Problems

Exponential Distributions deal with the length of the interval between events.
Intervals can be any dimension: time, length, area, volume. The following examples use time.

Probability of no events in interval x	use $P(X > x)$
Probability of the first event in interval x	use $P(X < x)$
Probability of at least one event in interval x	use $P(X < x)$
Probability of operating at least x hours	use $P(X > x)$
Probability of operating no less than x hours	use $P(X > x)$
Probability of operating for more than x hours	use $P(X > x)$
Probability of operating less than x hours	use $P(X < x)$
Probability of operating at most x hours	use $P(X < x)$
Probability of operating no more than x hours	use $P(X < x)$
Probability of failing in less than x hours	use $P(X < x)$
Probability of failing within x hours	use $P(X < x)$
Probability of failing in the next x hours	use $P(X < x)$
Probability of failing before x hours	use $P(X < x)$
Probability of failing after x hours	use $P(X > x)$
Probability of failing in more than x hours	use $P(X > x)$
Probability of not failing in the next x hours	use $P(X > x)$
Probability of not failing within the next x hours	use $P(X > x)$
Probability of not failing in less than x hours	use $P(X > x)$

$P(X > x) = e^{-\lambda x}$

$P(X < x) = 1 - e^{-\lambda x}$

$P(a < X < b) = e^{-\lambda a} - e^{-\lambda b}$

$P(X < (t_1 + t_2) \mid X > t_1) = P(X < t_2)$

$P(X > (t_1 + t_2) \mid X > t_1) = P(X > t_2)$

To find t, given $P(x < t) = y$ $\quad t = \dfrac{-\ln(1-y)}{\lambda}$

To find t, given $P(x > t) = y$ $\quad t = \dfrac{-\ln(y)}{\lambda}$

Mean Time Between Failures (MTBF) and Exponential Distributions

Many types of problems related to exponential distributions involve Mean Time Between Failure (MTBF). Suppose the MTBF for a certain electronic component is 2400 hours.
The ***Mean*** is 2400 hours. For an exponential distribution the ***Mean*** = $1/\lambda = 2400$, therefore $\lambda = 1/2400$

a. What is the probability that an electronic component will exceed 4000 before failing?

$P(X>x) = e^{-\lambda x}$ $\quad$ *with* $\lambda = 1/2400$ $\quad P(X>4000) = e^{-(1/2400)(4000)} = 0.1889$

b. What is the probability that an electronic component will fail before 1800 hours?

$P(X<x) = 1 - e^{-\lambda x}$ $\quad$ *with* $\lambda = 1/2400$ $\quad P(X<1800) = 1 - e^{-(1/2400)(1800)} = 1 - 0.4724 = 0.5276$

c. Determine the interval of time such that the probability of at least one failure is 50 %.

That is to say, find x such that $P(X<x) = 0.50 = 1 - e^{-\lambda x}$ $\qquad x = \dfrac{\ln(1-0.50)}{-1/2400} = 1664 \; hours$

d. Determine the interval of time such that the probability of no failures is 25 %.

That is to say, find x such that $P(X>x) = 0.25 = e^{-\lambda x}$ $\qquad x = \dfrac{\ln(0.25)}{-1/2400} = 3327 \; hours$

Exponential Memorylessness (Lack of Memory) Property

Given MTFB = 1200 hours.

Suppose an item has operated for 400 hours; what is the probability that it will fail in the next 600 hours?

What is the proper statement of the question?

$P(400 < X < 600)$?
$P(400 < X < 400 + 600)$?
$P(X < 400 + 600)$?
$P(X > 400) + P(X < 600)$?
$P(X > 400) + P(X > 600)$?
$P(X > 600)$?
$P(X < 600)$?

Let's look at a mathematical solution.

The statement is really a conditional probability question.
$P\{(x < 400 + 600) \mid (x > 400)\}$
That is to say, what is the probability that the item will fail within 400 + 600 hours,
given that it has already operated for 400 hours.
Or symbolically $P\{(x < t1 + t2) \mid (x > t1)\}$ where t1 = 400 and t2 = 600.

Remember $P(A \text{ AND } B) = P(A) \bullet P(B|A)$

$P(B|A) = P(A \text{ AND } B) / P(A)$

Let $A = (x > t1)$ $B = (x < t1 + t2)$

$A \text{ AND } B = (x > t1) \text{ AND } (x < t1 + t2)$

$P\{(x < t1 + t2) \mid (x > t1)\} = P\{(x > t1) \text{ AND } (x < t1 + t2)\} / P\{(x > t1)\}$

Note: $P\{(x > t1) \text{ AND } (x < t1 + t2)\} = P\{ t1 < x < t1 +t 2 \}$

Rewriting, we have
$P\{(x > t1) \text{ AND } (x < t1 + t2)\} / P\{(x > t1)\} = P\{ t1 < x < t1 + t2 \} / P\{(x > t1)\}$

$$P\{ t1 < x < t1 + t2 \} / P\{(x > t1)\} = \{e^{-\lambda(t1)} - e^{-\lambda(t1 + t2)}\} / e^{-\lambda(t1)}$$

$$= \{e^{-\lambda(t1)} - e^{-\lambda(t1)} e^{-\lambda(t2)}\} / e^{-\lambda(t1)}$$

$$= e^{-\lambda(t1)} \{1 - e^{-\lambda(t2)}\} / e^{-\lambda(t1)}$$

$$= 1 - e^{-\lambda(t2)} = P(x < t2) = P(X < 600)$$

Which is to say, it does not matter that the item had already operated for 400;
the question is simply what is the probability it will fail in the next 600 hours; i.e., $P(X < 600)$!