Alternating Current

Capacitive & Inductive Reactance and Complex Impedance
RC & RL Circuit Analyses (DC Transients, Time Constants, Steady State)

Electrical Theory (Alternating Current)

Charge $Q = C V$ Coulombs
Current $I = dQ/dt$ Amperes
Ohm's Law for AC $I_{RMS} = V_{RMS} / Z$ where Z is the Complex Impedance
$|Z| = [(R^2 + (XL - XC)^2]^{1/2}$
$\theta = \tan^{-1}[(XL - XC) / R]$
Power Factor $\cos\theta = R / Z$
Joule's Law Average Power $= ½ V_{peak} I_{peak} \cos\theta = V_{RMS} I_{RMS} \cos\theta$ Watts

$i = C\, dv/dt$ $v = 1/C \int i\, dt$
for $v = V_p \sin \omega t$ $i = C\, d(V_p \sin \omega t)/dt = \omega C V_p \cos \omega t = \omega C V_p \sin(\omega t + \pi/2)$

$v = L\, di/dt$ $i = 1/L \int v\, dt$
for $i = I_p \sin \omega t$ $v = L\, d(I_p \sin \omega t)/dt = \omega L I_p \cos \omega t = \omega L I_p \sin(\omega t + \pi/2)$

ELI the ICE man

Component	Voltage / Current
Resistor	In Phase
Capacitor	Lags
Inductor	Leads

Capacitive & Inductive Reactance and Complex Impedance

$\omega = 2\pi f$ $f = 0.159\omega$
Capacitive Reactance $X_C = 1 / \omega C = 1 / (2\pi f C) = 0.159 / f C$
Inductive Reactance $X_L = \omega L = 2\pi f L$
Complex Impedance
R in series with series CL $Z = R + j(2\pi f L - 1/(2\pi f C))$ Impedance is a minimum at resonance
R in series with parallel CL $Z = R + j(2\pi f L / (1 - (2\pi f)^2 LC))$ Impedance is a maximum at resonance

Time Constants
RC Circuit Time Constant = R C
RL Circuit Time Constant = L / R

Electrical Theory

Quantity	Symbol	Unit	Equation
Charge	Q	coulomb	$Q = \int i\,dt \quad Q = CV$
Current	I	ampere	$I = dQ/dt$
Voltage	V	volt	$V = dW/dQ$
Energy	W	joule	$W = \int V\,dQ = \int VI\,dt$
Power	P	watt	$P = dW/dt = IV$

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Electrical Theory - continued

Quantity	Symbol	Unit	Equation
Resistor	R	ohm	$V = IR$
Inductor	L	henry	$V = L\,dI/dt$ $I = 1/L \int V\,dt$
Capacitor	C	farad	$V = 1/C \int I\,dt$ $I = C\,dV/dt$

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Electrical Theory - continued

Ohm's Law

For DC: $I = V/R$ For AC: $I = V/Z$

Reactance (Resistance to AC)

Capacitive Reactance $X_C = 1 / j\omega C = -j/\omega C$

Inductive Reactance $X_L = j\omega L$

Series Impedance $= R + (X_L - X_C) = R + j(\omega L - 1/\omega C)$

$|R| = [\, R^2 + (X_L - X_C)^2 \,]^{1/2} \quad \theta = \mathrm{Tan}^{-1} [(X_L - X_C) / R]$

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Series and Parallel Components

Component	**Series**	**Parallel**
R	$R_{Eq} = R_1 + R_2 + R_3$	$1/R_{Eq} = 1/R_1 + 1/R_2 + 1/R_3$
Z	$Z_{Eq} = Z_1 + Z_2 + Z_3$	$1/R_{Eq} = 1/Z_1 + 1/Z_2 + 1/Z_3$
L	$L_{Eq} = L_1 + L_2 + L_3$	$1/L_{Eq} = 1/L_1 + 1/L_2 + 1/L_3$

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DC Circuit Components

Component	Impedance	Current	Power/Energy
R	R	$I = V/R$	$I^2 R$
L	Zero	Infinite	$1/2\, LI^2$
C	Infinite	Zero	$1/2\, CV^2$

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AC Sinusoidal Analysis

Resistor	R	$I = V/R$	$P = I^2R = V^2/R$
Inductor	$X_L = j\omega L$	$I = -jV_L/\omega L$	$Q_L = I^2X_L = V_L^2/X_L$
Capacitor	$X_C = -j/\omega C$	$I = jV_C\omega C$	$Q_C = I^2X_C = V_C^2/X_C$

Current $I = I_R + j\,I_X$

Voltage $V = V_R + j\,V_X$

Complex Power $S = VI^* = (V_R + j\,V_X)(I_R - j\,I_X)$

Complex Power $S = P + j\,Q$

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RESISTOR, INDUCTOR, CAPACITOR

When electrical energy is supplied to a circuit element, it will respond in one or more of the following three ways. If the energy is consumed, then the circuit element is a pure *resistor*. If the energy is stored in a magnetic field, the element is a pure *inductor*. And if the energy is stored in an electric field, the element is a pure *capacitor*. A practical circuit device exhibits more than one of the above and perhaps all three at the same time, but one may be predominant. A coil may be designed to have a high inductance, but the wire with which it is wound has some resistance; hence the coil has both properties.

RESISTANCE R

The potential difference $v(t)$ across the terminals of a pure resistor is directly proportional to the current $i(t)$ in it. The constant of proportionality R is called the resistance of the resistor and is expressed in volts/ampere or ohms.

$$v(t) = R\,i(t) \quad \text{and} \quad i(t) = \frac{v(t)}{R}$$

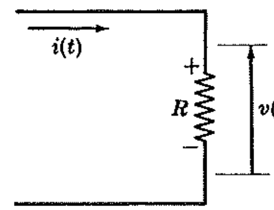


Fig. 1-3

No restriction is placed on $v(t)$ and $i(t)$; they may be constant with respect to time, as in D.C. circuits, or they may be sine or cosine functions, etc.

Lower case letters (v, i, p) indicate general functions of time. Capital letters (V, I, P) denote constant quantities, and peak or maximum values carry a subscript (V_m, I_m, P_m).

INDUCTANCE L

When the current in a circuit is changing, the magnetic flux linking the same circuit changes. This change in flux causes an emf v to be induced in the circuit. The induced emf v is proportional to the time rate of change of current if the permeability is constant. The constant of proportionality is called the *self-inductance* or *inductance* of the circuit.

$$v(t) = L\frac{di}{dt} \quad \text{and} \quad i(t) = \frac{1}{L}\int v\,dt$$

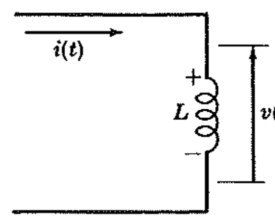


Fig. 1-4

When v is in volts and di/dt in amperes/sec, L is in volt-sec/ampere or *henries*. The self-inductance of a circuit is 1 henry (1 h) if an emf of 1 volt is induced in it when the current changes at the rate of 1 ampere/sec.

CAPACITANCE C

The potential difference v between the terminals of a capacitor is proportional to the charge q on it. The constant of proportionality C is called the *capacitance* of the capacitor.

$$q(t) = C\,v(t), \quad i = \frac{dq}{dt} = C\frac{dv}{dt}, \quad v(t) = \frac{1}{C}\int i\,dt$$

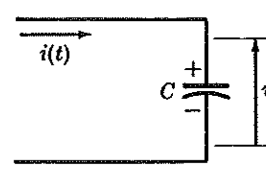


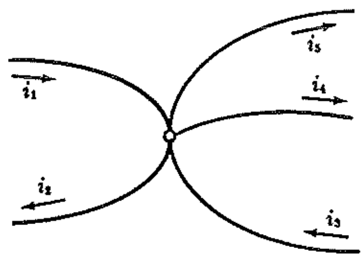
Fig. 1-5

When q is in coulombs and v in volts, C is in coulombs/volt or *farads*. A capacitor has capacitance 1 farad (1 f) if it requires 1 coulomb of charge per volt of potential difference between its conductors. Convenient submultiples of the farad are

$$1\,\mu\text{f} = 1\text{ microfarad} = 10^{-6}\text{ f} \quad \text{and} \quad 1\,\mu\mu\text{f} = 1\text{ micromicrofarad} = 10^{-12}\text{ f}$$

KIRCHHOFF'S LAWS

1. The sum of the currents entering a junction is equal to the sum of the currents leaving the junction. If the currents toward a junction are considered positive and those away from the same junction negative, then this law states that the algebraic sum of all the currents meeting at a common junction is zero.

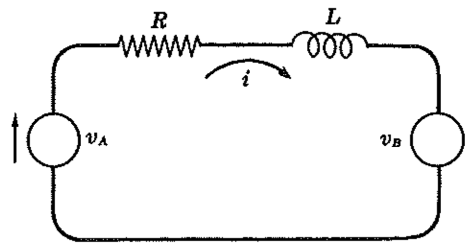


Σ currents entering $=$ Σ currents leaving

$$i_1 + i_3 = i_2 + i_4 + i_5$$

or $$i_1 + i_3 - i_2 - i_4 - i_5 = 0$$

Fig. 1-6



Σ potential rises $=$ Σ potential drops

$$v_A - v_B = Ri + L(di/dt)$$

or $$v_A - v_B - Ri - L(di/dt) = 0$$

Fig. 1-7

2. The sum of the rises of potential around any closed circuit equals the sum of the drops of potential in that circuit. In other words, the algebraic sum of the potential differences around a closed circuit is zero. With more than one source when the directions do not agree, the voltage of the source is taken as positive if it is in the direction of the assumed current.

Circuit Response of Single Elements

Element	Voltage across element	Current in element
Resistance R	$v(t) = R\,i(t)$	$i(t) = \frac{v(t)}{R}$
Inductance L	$v(t) = L\frac{di}{dt}$	$i(t) = \frac{1}{L}\int v\,dt$
Capacitance C	$v(t) = \frac{1}{C}\int i\,dt$	$i(t) = C\frac{dv}{dt}$

Units in the MKS System

Quantity		Unit		Quantity		Unit	
Length	l	meter	m	Charge	Q, q	coulomb	c
Mass	m	kilogram	kg	Potential	V, v	volt	v
Time	t	second	sec	Current	I, i	ampere	amp
Force	F, f	newton	nt	Resistance	R	ohm	Ω
Energy	W, w	joule	j	Inductance	L	henry	h
Power	P, p	watt	w	Capacitance	C	farad	f

Equations and Relationships

Inductive Reactance $X_L = 2\pi\, f\, L$

Capacitive Reactance $X_C = \dfrac{1}{2\pi\, f\, C}$

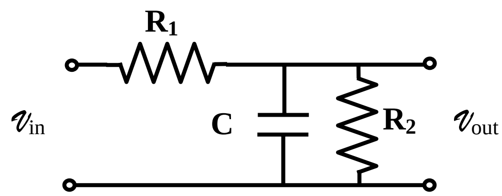
	RC Circuit	RL Circuit	RCL Circuit
Cut-off Frequency Resonant Frequency	$f_0 = \dfrac{1}{2\pi\, RC}$	$f_0 = \dfrac{1}{2\pi\, L/R}$	$f_0 = \dfrac{1}{2\pi\sqrt{LC}}$
Time Constant	$t = RC$	$t = L/R$	$t = \dfrac{R\sqrt{C/L}}{2}$

RCL Series Impedance $Z = \sqrt{R^2 + (X_L - X_C)^2}$

R&CL Parallel Impedance $Z = \sqrt{R^2 + \left(\dfrac{X_L\, X_C}{X_L - X_C}\right)^2}$

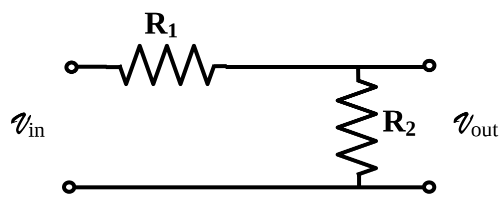
RCL Parallel Impedance $Z = \dfrac{R\, X_L X_C}{X_L X_C - R(X_L - X_C)}$

Common Configuration

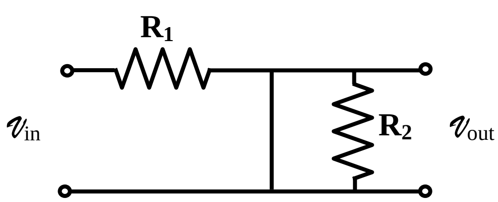


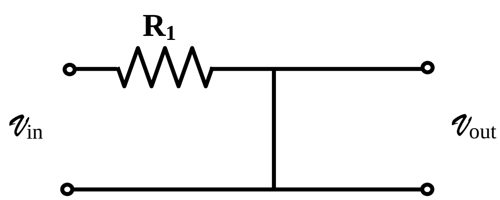
Notes:

When ω = 0, $X_C \rightarrow \infty$, i.e., **C** appears as an open circuit, so that $V_{out} = \dfrac{R_2}{R_1 + R_2}$



When ω >> 0, $X_C = 0$, i.e., **C** appears as a short circuit, so that $V_{out} = 0$





Understanding the Behavior of Complex Impedances

Understanding the Behavior of Complex Impedances at very low frequency (i.e., $f \approx 0$) and at very high frequency (i.e., $f >> 0$ or $f \approx \infty$).

For Inductive Reactance, $X_L = \omega L$
If $\omega = 0$, then $X_L = \omega L = 0$
If $\omega >> 0$, then $X_L = \omega L >> 0 \approx \infty$

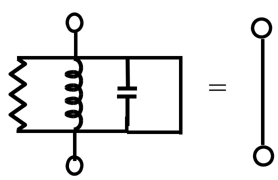
For Capacitive Reactance, $X_C = 1 / \omega C$
If $\omega = 0$, then $X_C = 1 / \omega C >> 0 \approx \infty$
If $\omega >> 0$, then $X_C = 1 / \omega C \approx 0$

R = R regardless of ω.

From Ohm's Law for Alternating Current, Impedance $Z = v(t) / i(t)$
$Z = R + j(X)$ where X is the reactant component due to circuit capacitors and inductors at a given frequency.

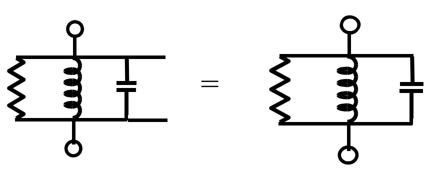
IF $Z = 0$, then **Short**, i.e., like a wire conductor, very high current ($I = V/Z$).
If $Z >> 0$ ($Z \approx \infty$), then **Open**, i.e., like a open switch, no current.

A **Short** in parallel with any other number of elements appears as a Short overall; i.e., just a wire conductor.



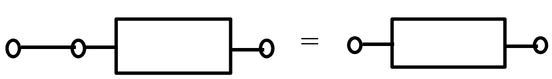
Short

An **Open** in parallel with another element can be considered to be no existent, i.e., no effect.



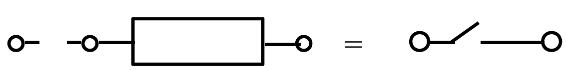
No Effect

A **Short** in series with another element can be considered to be just a wire conductor.



No Effect

An **Open** in series with another number of elements appears as an Open Switch.

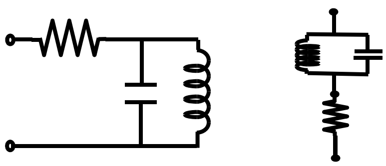


Open Switch

If $\omega = 0$, then $X_L = \omega L = 0$ and $X_C = 1 / \omega C \gg 0$.
If $\omega \gg 0$, then $X_L = \omega L \gg 0$ and $X_C = 1/ \omega C = 0$.
R = R regardless of ω.
IF Z = 0, then Short, i.e., like a wire conductor, very high current (I = V/R).
If Z >> 0 ($Z \approx \infty$), then Open, i.e., like a open switch, no current.

For series: $Z + 0 = Z$ and $Z + \infty = \infty$ (Open)
For parallel, $Z \parallel 0 = \frac{Z(0)}{Z+0} = 0$ (Short) and $Z \parallel \infty = \frac{Z(\infty)}{Z+\infty} = \frac{Z(\infty)}{\infty} = Z$

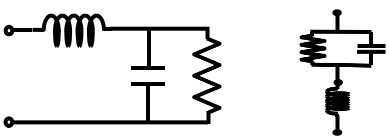
From Complex Impedance Quiz & BME/ISE 3511Fall 2015 Test Four

A. 

$$R + \frac{X_C X_L}{X_C + X_L}$$

if $\boldsymbol{\omega = 0}$, then $R + \frac{X_C X_L}{X_C + X_L} \approx R + 0 = R$ (Note: $X_L = 0$) **Overall Effect = Resistive**

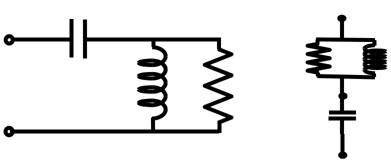
if $\boldsymbol{\omega \gg 0}$, then $R + \frac{X_C X_L}{X_C + X_L} \approx R + 0 = R$ (Note: $X_C = 0$) **Overall Effect = Resistive**

B. 

$$X_L + \frac{R X_C}{R + X_C}$$

if $\boldsymbol{\omega = 0}$, then $X_L + \frac{R X_C}{R + X_C} \approx 0 + \frac{R X_C}{R + X_C} \approx 0 + \frac{R X_C}{X_C} = 0 + R = R$ (Note: $X_L = 0$ and $X_C \gg 0$)
Overall Effect = Resistive

if $\omega \gg \mathbf{0}$, then $X_L + \frac{R X_C}{R + X_C} = \infty + 0 = \infty$ Open (Note: $X_L = \infty$ and $X_C = 0$)
Overall Effect = Open

C. 

$$X_C + \frac{R X_L}{R + X_L}$$

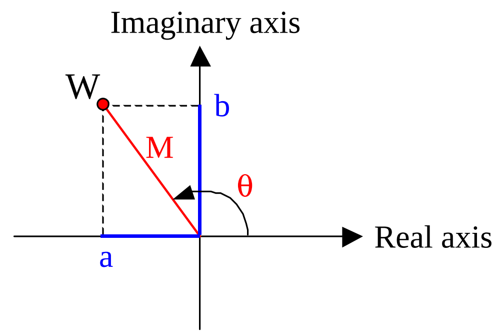
if $\boldsymbol{\omega = 0}$, then $X_C + \frac{R X_L}{R + X_L} \approx \infty + \frac{R(0)}{R+0} \approx \infty + 0 = \infty$ Open (Note: $X_C \gg 0$ and $X_L = 0$)
Overall Effect = Open

if $\boldsymbol{\omega \gg 0}$, then $X_C + \frac{R X_L}{R + X_L} \approx 0 + \frac{R X_L}{R + X_L} \approx 0 + \frac{R X_L}{X_L} = 0 + R = R$ (Note: $X_C = 0$ and $X_L \gg 0$)
Overall Effect = Resistive

Review of Complex Numbers

A complex number represents a point in a 2D space. The value of the complex number can be represented either by its real part (a) and imaginary part (b), or by its magnitude (M) and its phase angle (θ), as shown in the figure below.

$$\mathbf{W} = a + jb$$
$$= M\, e^{jq} = (M \cos q) + j(M \sin q)$$



Computations involving two complex numbers: x = a + jb = $M\, e^{jq}$ and y = c + jd = $N\, e^{jj}$

(a) Addition and subtraction:

z = x ± y = (a + c) + j (b + d)

(b) Multiplication:

z = x y = (a + jb) (c + jd) = (ac – bd) + j (bc + ad)

z = x y = $(M\, e^{jq})(N e^{jj}) = (MN)\, e^{j(q+j)}$

Therefore, Magnitude of z = (magnitude of x) · (magnitude of y)
Phase angle of z = (phase angle of x) + (phase angle of y)

(c) Division:

$$z = \frac{x}{y} = \frac{a + jb}{c + jd} = \frac{(a + jb)(c - jd)}{(c + jd)(c - jd)} = \left(\frac{ac + bd}{c^2 + d^2}\right) + j\left(\frac{bc - ad}{c^2 + d^2}\right)$$

$$z = \frac{x}{y} = \frac{M\, e^{jq}}{N\, e^{jj}} = \left(\frac{M}{N}\right) e^{j(q - j)}$$

Therefore, Magnitude of z = (magnitude of x) ÷ (magnitude of y)
Phase angle of z = (phase angle of x) - (phase angle of y)