State Space Description of Dynamic Systems

- A modern complex system may have many inputs and many outputs, and these may be interrelated in a complicated manner.
- To analyze such a system, it is essential to reduce the complexity of the mathematical expressions.
- The state-space approach to system analysis is best suited from this viewpoint.
- While conventional control theory is based on the input-output relationship or transfer function, modern control theory is based on the description of system equations in terms of $n$ first-order differential equations. These equations may be combined into a first-order vector-matrix differential equations.
- The use of vector-matrix notation greatly simplifies the mathematical representation of systems of equations.
- The increase in the number of state-variables, the number of inputs, or the number of outputs does not increase the complexity of the equations. In fact, the analysis of complicated multi-input multi-output systems can be carried out by the procedures that are only slightly more complicated than those required for the analysis of systems of first-order scalar differential equations.
Consider a SISO system defined by a second-order differential equation

\[ \ddot{y} + a_1 \dot{y} + a_2 y = u \]  

(1)

where \( u \) is the input and \( y \) is the output. The transfer function of the system is given by

\[ \frac{Y(s)}{U(s)} = \frac{1}{s^2 + a_1 s + a_2} \]

Solving for the highest derivative, \( \ddot{y} \), in equation (1), we get

\[ \ddot{y} = u - a_1 \dot{y} - a_2 y \]

Integrating \( \ddot{y} \) twice, both \( \dot{y} \) and \( y \) are obtained as shown in Figure 1a. The loop is then closed by satisfying the requirement of the differential equation as shown in Figure 1b.

Figure 1. Analog computer diagram for the system.

The convenient choices for the state variables are the output of the integrators, i.e., \( y \) and \( \dot{y} \).

Let \( x_1 = y \)

and \( x_2 = \dot{y} \)
Therefore,

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -a_2 x_1 - a_1 x_2 + u
\end{align*}
\]

The vector matrix representation for the system is

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
-a_2 & -a_1 
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 
\end{bmatrix} +
\begin{bmatrix}
0 \\
1 
\end{bmatrix} u
\]

\[
y = \begin{bmatrix}
1 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 
\end{bmatrix} + [0] u
\]

Thus, if the system is described by a set of linear differential equations, the state equations can be written as

\[
\dot{x}(t) = A x(t) + B u(t)
\]

\[
y(t) = C x(t) + D u(t)
\]

Where \(A, B, C\) and \(D\) are matrices and

\[
x = \begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n 
\end{bmatrix}, \quad u = \begin{bmatrix}
\dot{u}_1 \\
\dot{u}_2 \\
\vdots \\
\dot{u}_m 
\end{bmatrix} \quad \text{and} \quad y = \begin{bmatrix}
\dot{y}_1 \\
\dot{y}_2 \\
\vdots \\
\dot{y}_p 
\end{bmatrix}
\]

For this example,

\[
A = \begin{bmatrix}
0 & 1 \\
-a_2 & -a_1 
\end{bmatrix}, \quad B = \begin{bmatrix}
0 \\
1 
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
1 & 0
\end{bmatrix} \quad \text{and} \quad D = [0]
\]

This form of state-space representation is called the Controllable Canonical form or CCF. The other form is called the Observably Canonical form or OCF. Now, let us try to obtain the OCF of the system represented by equation (1).
\[
\frac{Y(s)}{U(s)} = \frac{1}{s^2 + a_1 s + a_2} = \frac{s^2}{1 + a_1 s + a_2 s^2}
\]

or \[(1 + a_1 s + a_2 s^2) Y(s) = s^2 U(s)\]

or \[Y(s) = -(a_1 s + a_2 s^3) Y(s) + s^2 U(s)\]

(2)

Figure 2 shows the state diagram that results from equation (2). The output of the integrators are designated as the state variables. However, unlike the usual convention, the state variables are assigned in descending order from right to left.

Figure 2. OCF state diagram of the system in eqn. (1).

Therefore,

\[
\begin{align*}
\dot{x}_1 &= -a_2 x_2 + u \\
\dot{x}_2 &= x_1 - a_1 x_2
\end{align*}
\]

The vector matrix representation for the system is

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} =
\begin{bmatrix}
0 & -a_2 \\
1 & -a_1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} +
\begin{bmatrix}
1 \\
0
\end{bmatrix} u
\]

\[
y =
\begin{bmatrix}
0 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} + [0] u
\]

This form of state-space representation is called the Observably Canonical form of OCF.
Now, let us consider a SISO system defined by a second-order differential equation
\[ y + a_1 y + a_2 y = b_1 u + b_2 u \]
whose transfer function is given by
\[ \frac{Y(s)}{U(s)} = \frac{(b_1 s + b_2)}{s^2 + a_1 s + a_2} \]
Multiply the numerator and the denominator of the transfer function by a dummy variable \( x(s) \), we get
\[ \frac{Y(s)}{U(s)} = \frac{(b_1 s + b_2) x(s)}{(s^2 + a_1 s + a_2) x(s)} \]
Equating the numerators and denominators on both sides, we get
\[ Y(s) = (b_1 s + b_2) x(s) \]
and \( U(s) = (s^2 + a_1 s + a_2) x(s) \)
\[ \therefore \quad x(s) = U(s) - a_1 x(s) - a_2 x(s) \]
\[ \therefore \quad \dot{x}_1 = \dot{x}_2 \]
\[ \dot{x}_2 = -a_2 x_1 - a_1 x_2 + u \]
\[ \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \]
\[ y = \begin{bmatrix} b_2 \\ b_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0] u \]
When a SISO system is defined by a transfer function that is not proper,

\[ \frac{Y(s)}{U(s)} = \frac{b_0 s^2 + c_1 s + c_2}{s^2 + a_1 s + a_2} \]

First, the numerator is divided by the denominator to make the transfer function proper, i.e.

\[ \frac{Y(s)}{U(s)} = b_0 + \frac{b_1 s + b_2}{s^2 + a_1 s + a_2} \]

where \( b_1 = c_1 - a_1 b_0 \) and \( b_2 = c_2 - a_2 b_0 \).

The state-space representation of this system can be written as

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
-a_2 & -a_1
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} + \begin{bmatrix}
0 \\
1
\end{bmatrix} u
\]

\[ y = \begin{bmatrix}
b_2 & b_1
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} + b_0 u \]

**Example 1:** Obtain CCF and OCF for the following input-output transfer function

\[ \frac{Y(s)}{U(s)} = \frac{2s^2 + s + 5}{s^3 + 6s^2 + 11s + 4} \]

**CCF:**

\[ Y(s) = \frac{(2s^2 + s + 5)X(s)}{s^3 + 6s^2 + 11s + 4} \]

\[ U(s) = \frac{(s^3 + 6s^2 + 11s + 4)X(s)}{s^3 + 6s^2 + 11s + 4} \]

\[ sX(s) = U(s) - 6s^2X(s) - 11sX(s) - 4X(s) \]
The state-space equations of the system in CCF are

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-4 & -11 & -6
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix} u
\]

\[y = \begin{bmatrix}
5 & 1 & 2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} + [0] u
\]

\[\text{OCF:} \quad y(s) = \left(2s^{-1} + s^2 + 5s^3\right)u(s) - \left(6s^{-1} + 11s^2 + 4s^3\right)y(s)
\]

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & -4 \\
1 & 0 & -11 \\
0 & 1 & -6
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} +
\begin{bmatrix}
5 \\
1 \\
2
\end{bmatrix} u
\]

\[y = \begin{bmatrix}
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} + [0] u
\]
Other Form of Decomposition

Cascade Decomposition

Cascade decomposition refers to transfer functions that are written as products of simple first-order or second-order components. Consider the following transfer function, which is the product of two first-order transfer functions:

\[
\frac{Y(s)}{U(s)} = K \left( \frac{s+b_1}{s+a_1} \right) \left( \frac{s+b_2}{s+a_2} \right)
\]

where \(a_1, a_2, b_1\), and \(b_2\) are real constants. Each of first-order transfer functions is decomposed by the direct decomposition (CCF) and the two state diagrams are connected in cascade as shown in the figure.

The state-space representation of the system can now be written as:

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix}
-a_1 & b_2-a_2 \\
0 & -a_2
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} + \begin{bmatrix}
K
\end{bmatrix} u
\]

\[
y = \begin{bmatrix}
b_1-a_1 & b_2-a_2
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} + K u
\]
When the overall transfer function has complex poles or zeros, the individual factors related to these poles or zeros should be in second order form. As an example, consider the following transfer function

\[
\frac{Y(s)}{U(s)} = \left(\frac{s+5}{s+2}\right)\left(\frac{s+15}{s^2+3s+4}\right)
\]

where the poles of the second term are complex.

The state diagram of the system with the two subsystems connected in cascade is shown below.

The state-space representation of the system can now be written as

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 \\
-4 & -3 & 3 \\
0 & 0 & -2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
+ \begin{bmatrix}
0 \\
1 \\
1
\end{bmatrix} u
\]

\[
y = \begin{bmatrix}
15 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} + \begin{bmatrix}
0
\end{bmatrix} u
\]
Parallel Decomposition

- When the denominator of the transfer function is in factored form, the transfer function may be expanded by partial-fraction expansion.
- The resulting state diagram will consist of simple first- or second-order systems connected in parallel, which leads to the state equations in Diagonal Canonical Form (DCF) or Jordan Canonical Form (JCF), the latter in the case of multiple-order eigenvalues.
- Consider that a second-order system is represented by the transfer function

\[
\frac{Y(s)}{U(s)} = \frac{Q(s)}{(s+a_1)(s+a_2)}
\]

where \( Q(s) \) is a polynomial of order less than 2, and \( a_1 \) and \( a_2 \) are real and distinct.
- The partial fractions of the transfer function can be written as

\[
\frac{Y(s)}{U(s)} = \frac{K_1}{s+a_1} + \frac{K_2}{s+a_2}
\]

where \( K_1 \) and \( K_2 \) are real constants.
- The state diagram of the system is drawn by the parallel combination of the state diagram of each of the first-order terms as shown below.

\[ U \rightarrow x_1 \rightarrow y \]
\[ U \rightarrow x_2 \rightarrow y \]
\[ x_1 = a_1 x_1 + K_1 \]
\[ x_2 = a_2 x_2 + K_2 \]

Diagram:

```
  U  \( x_1 \) \( x_2 \) \( y \)
  ^           ^
  \( x_1 \)   \( x_2 \)
  \( a_1 \)  \( a_2 \)
  \( K_1 \)  \( K_2 \)
```
The state-space representation of the system can now be written as

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} =
\begin{bmatrix}
-a_1 & 0 \\
0 & -a_2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} +
\begin{bmatrix}
1 \\
1
\end{bmatrix} u
\]

\[
y =
\begin{bmatrix}
k_1 & k_2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} + [0] u
\]

The conclusion is that for transfer functions with distinct poles, parallel decomposition will lead to the Diagonal Canonical Form (DCF) for the state equations.

For transfer functions with multiple-order eigenvalues, parallel decomposition to a state diagram with a minimum number of integrators will lead to the Jordan Canonical Form (JCF) state equations. This is clarified with example 2.

**Example 2**: Consider the following transfer function

\[
\frac{Y(s)}{U(s)} = \frac{2s^2 + 6s + 5}{(s+1)^2(s+2)}
\]

\[= \frac{A}{(s+1)^2} + \frac{B}{s+1} + \frac{C}{s+2}\]

\[A = \frac{2s^2 + 6s + 5}{s+2} \bigg|_{s=-1} = \frac{1}{1} = 1\]

\[C = \frac{2s^2 + 6s + 5}{(s+1)^2} \bigg|_{s=-2} = 1\]

and \(2s^2 + 6s + 5 = A(s+2) + B(s^2 + 3s + 2) + C(s + 2s + 1)\)

\[s^2: \quad a = B + C \quad \Rightarrow \quad B = a - c = 1\]
\[
\frac{y(s)}{u(s)} = \frac{1}{(s+1)^2} + \frac{1}{s+1} + \frac{1}{s+2}
\]

- The state diagram of the system is shown below.

- The state-space representation of the system can now be written as

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{bmatrix} =
\begin{bmatrix}
-1 & 1 & 0 \\
0 & -1 & 0 \\
0 & 0 & -2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} +
\begin{bmatrix}
0 \\
1 \\
1
\end{bmatrix} u
\]

\[
y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\
x_2 \\
x_3 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} u
\]
Consider that a multivariable system is described by the differential equations:

\[
\frac{d^2 y_1(t)}{dt^2} + 4 \frac{dy_1(t)}{dt} - 3 y_2(t) = u_1(t) + 2 u_2(t)
\]

\[
\frac{dy_1(t)}{dt} + \frac{dy_2(t)}{dt} + y_1(t) + 2 y_2(t) = u_2(t)
\]

\[
\dot{y}_1 = -4 \dot{y}_1 + 3 y_2 + u_1 + 2 u_2
\]

\[
\dot{y}_2 = -\dot{y}_1 + y_1 - 2 y_2 + u_2
\]

\[
\dot{x}_1 = x_2
\]

\[
\dot{x}_2 = -4 x_2 + 3 x_3 + u_1 + 2 u_2
\]

\[
\dot{x}_3 = -x_1 - x_2 - 2 x_3 + u_2
\]

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 \\
0 & -4 & 3 \\
-1 & -1 & -2 
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 
\end{bmatrix} +
\begin{bmatrix}
0 & 0 \\
1 & 2 \\
0 & 1 
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 
\end{bmatrix}
\]

\[
\begin{bmatrix}
\dot{y}_1 \\
\dot{y}_2 
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 
\end{bmatrix}
\]
In general, linear constant coefficient continuous-time state equations are of the form

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\vdots \\
\dot{x}_n 
\end{bmatrix} = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n 
\end{bmatrix} + \begin{bmatrix}
b_{11} & b_{12} & \cdots & b_{1r} \\
b_{21} & b_{22} & \cdots & b_{2r} \\
\vdots & \vdots & \ddots & \vdots \\
b_{m1} & b_{m2} & \cdots & b_{mr}
\end{bmatrix} \begin{bmatrix}
u_1 \\
u_2 \\
\vdots \\
u_r 
\end{bmatrix}
\]

or

\[
\dot{x} = Ax + Bu
\]

where the state vector \( x \) is an \( n \)-vector, the state coupling matrix \( A \) is \( nxn \), the input vector \( u \) is an \( r \)-vector of input signals, and the input coupling matrix \( B \) is \( nxr \). The order of the system is the dimension of the state vector \( n \). The general form of the output equation is

\[
\begin{bmatrix}
y_1 \\
y_2 \\
\vdots \\
y_m
\end{bmatrix} = \begin{bmatrix}
c_{11} & c_{12} & \cdots & c_{1n} \\
c_{21} & c_{22} & \cdots & c_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
c_{m1} & c_{m2} & \cdots & c_{mn}
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n 
\end{bmatrix} + \begin{bmatrix}
d_{11} & d_{12} & \cdots & d_{1r} \\
d_{21} & d_{22} & \cdots & d_{2r} \\
\vdots & \vdots & \ddots & \vdots \\
d_{m1} & d_{m2} & \cdots & d_{mr}
\end{bmatrix} \begin{bmatrix}
u_1 \\
u_2 \\
\vdots \\
u_r 
\end{bmatrix}
\]

or

\[
y = Cx + Du
\]

where the output vector \( y \) is an \( m \)-vector of output signals, the output coupling matrix \( C \) is \( mxn \), and the input-to-output coupling matrix \( D \) is \( mxr \).
A block diagram that shows how the input, output and state vectors are related in general in a continuous-time state variable system is shown in the Figure 3. The wide arrows in this figure represent vectors of signals.

Figure 3. Block diagram showing the relations between signal vectors in a continuous-time state variable model.
Time Domain Solution - State Transition Matrix

- Once the state equations of a linear time-invariant system are expressed in the state-space form, the next step often involves solving the equations given the initial state vector $x(t_0)$ and the input $u(t)$ for $t > t_0$. The first part of the state-space equations on the right-hand side is known as the homogeneous part of the state equation, and the last term represents the forcing function $u(t)$.

- Let us first analyze a first-order system with the state equation

$$\frac{dx}{dt} = ax(t) + bu(t)$$

- The solution can be obtained by multiplying both sides of the equation by the "integrating factor" $e^{-at}$ as

$$e^{-at}\frac{dx}{dt} - ae^{-at}x(t) = e^{-at}bu(t)$$

$$\Rightarrow \frac{d}{dt}(e^{-at}x(t)) = e^{-at}bu(t)$$

Integrating both sides gives

$$e^{-at}x(t) = \int e^{-at}bu(t)dt + \text{(arbitrary constant)}$$

- If the integration is begun at $t=0$, then

$$e^{-at}x(t) = \int_0^t e^{-a(t-\tau)}bu(\tau)d\tau + x(0^-), \quad t > 0$$
Solving for \( x(t) \),

\[
x(t) = e^{at} x(0^-) + \int_0^t e^{a(t-\tau)} b u(\tau) \, d\tau, \quad t \geq 0
\]

- The integral is the convolution of the function \( e^{at} \) and the input term \( b u(t) \).

- In terms of Laplace transform, the state of the first-order system is given by

\[
S \, x(s) - x(0^-) = a \, x(s) + b \, u(s)
\]

\[
\therefore \quad x(s) = \frac{1}{s-a} x(0^-) + \frac{1}{s-a} b \, u(s)
\]

\[
x(t) = \mathcal{L}^{-1} \left[ \frac{1}{s-a} x(0^-) + \frac{1}{s-a} b \, u(s) \right]
\]

\[
= e^{at} x(0^-) + \mathcal{L}^{-1} \left[ \frac{1}{s-a} b \, u(s) \right], \quad t \geq 0
\]

\[
= e^{at} x(0^-) + \int_0^t e^{a(t-\tau)} b \, u(\tau) \, d\tau, \quad t \geq 0
\]

- For digital computation, convolution using numerical integration is generally easier than dealing with Laplace transform.

- Now, let us analyze a \( n \)th order system (a system described by \( n \) first order differential equation in the form of state-space) described by

\[
\dot{x}(t) = A \, x(t) + B \, u(t)
\]

\[
y(t) = C \, x(t) + D \, u(t)
\]
Taking the Laplace transform of state and output equations of a state variable system model, we get

\[ sX(s) - X(0-) = AX(s) + BU(s) \]
\[ Y(s) = CX(s) + DU(s) \]

- Solving for the transform of the state vector gives

\[(SI-A)X(s) = X(0-) + BU(s)\]

\[X(s) = (SI-A)^{-1}X(0-) + (SI-A)^{-1}BU(s)\]

\[\text{Zero-input component of state vector} \quad \text{Zero-state component of state vector}\]

- The zero-input component is the solution when the input is zero, i.e., state due to initial conditions.
- The zero-state component of the solution when the initial conditions are zero, i.e., component due to input.
- The transform of the system output is then

\[Y(s) = C(SI-A)^{-1}X(0-) + [C(SI-A)^{-1}B + D]U(s)\]

**Example 3:** Consider a second-order SISO system described by

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} =
\begin{bmatrix}
-7 & 1 \\
-12 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} +
\begin{bmatrix}
2 \\
-1
\end{bmatrix}u(t)
\]

\[y =
\begin{bmatrix}
3 & -4
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} +
\begin{bmatrix}
-2
\end{bmatrix}u(t)\]

Let the system input be \(u(t) = 3e^t, \quad t \geq 0\)

and the initial system state be

\[
\begin{bmatrix}
x_{1}(0-) \\
x_{2}(0-)
\end{bmatrix} =
\begin{bmatrix}
-6 \\
1
\end{bmatrix}
\]
\[ X(s) = \begin{bmatrix} x_1(s) \\ x_2(s) \end{bmatrix} = \begin{bmatrix} s+7 & -1 \\ 12 & s \end{bmatrix}^{-1} \begin{bmatrix} -6 \\ 1 \end{bmatrix} + \begin{bmatrix} s+7 & -1 \\ 12 & s \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 8+1 \end{bmatrix} = \begin{bmatrix} -6s^2 + s - 2 \\ \frac{s^2 + 17s - 14}{(s+1)(s+3)(s+4)} \end{bmatrix} \]

\[ x_1(t) = L^{-1} x_1(s) \text{ and } x_2(t) = L^{-1} x_2(s) \]

\[ Y(s) = \begin{bmatrix} 3 & -4 \end{bmatrix} \begin{bmatrix} -6s^2 + s - 2 \\ \frac{s^2 + 17s - 14}{(s+1)(s+3)(s+4)} \end{bmatrix} + \begin{bmatrix} -6 \\ s+1 \end{bmatrix} = \frac{-28s^2 - 347s - 22}{(s+1)(s+3)(s+4)} \]

\[ = \frac{306}{s+4} - \frac{383.5}{s+3} + \frac{49.5}{s+1} \]

\[ y(t) = (306e^{-4t} - 383.5e^{-3t} + 49.5e^{-t}), \quad t \geq 0 \]
Determination of Transfer Functions from State Variable

- The transfer functions of a state variable system are found by expressing the Laplace transform of the output vector in terms of the Laplace transform of the input vector when the initial condition is zero, i.e.

\[ H(s) = \frac{Y(s)}{U(s)} \quad | \quad x(0-) = 0 \]

From the previous equation

\[ Y(s) = C(sI-A)^{-1}x(0-) + [C(sI-A)^{-1}B+D]U(s) \]

Substituting \( x(0-) = 0 \), we get

\[ \frac{Y(s)}{U(s)} = H(s) = C(sI-A)^{-1}B+D \]

- \( H(s) \) is an \( m \times r \) matrix, where \( m \) is the no. of outputs and \( r \) is the no. of inputs in \( u(t) \).

- The elements of \( H(s) \) are function of the variable \( s \) and the element in the \( i \)th row and \( j \)th column of \( H(s) \) is the transfer function relating the \( i \)th output and \( j \)th input,

\[ H_{ij}(s) = \frac{Y_i(s)}{U_j(s)} \quad | \quad \text{Zero ICs and all other inputs are zero} \]

- Because \( (sI-A)^{-1} = \text{adj}(sI-A) / |sI-A| \), the \( n \)th order characteristic polynomial \( |sI-A| \) always occurs as the denominator polynomial of each individual transfer function.
- The roots of the characteristic polynomial are the poles of the system transfer functions.

- The system's poles are also called the eigenvalues because the solution of the CP are the eigenvalues of matrix A.

- The system is stable if all the eigenvalues of A are in the left half of the s-plane.

**Example 4:** The three inputs/two outputs, second-order system is given by

\[
\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t)
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
-5 & -2
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix} +
\begin{bmatrix}
1 & 0 & -1 \\
0 & 2 & 3
\end{bmatrix}
\begin{bmatrix}
u_1(t) \\
u_2(t) \\
u_3(t)
\end{bmatrix}
\]

\[
\begin{bmatrix}
y_1(t) \\
y_2(t)
\end{bmatrix} =
\begin{bmatrix}
0 & -3 \\
2 & -4
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix} +
\begin{bmatrix}
1 & -2 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
u_1(t) \\
u_2(t) \\
u_3(t)
\end{bmatrix}
\]

\[H(s) = C(SI-A)^{-1}B + D\]

\[(SI-A) = \begin{bmatrix}
S & -1 \\
5 & S + 2
\end{bmatrix}\]

\[(SI-A)^{-1} = \frac{1}{S^2 + 2S + 5} \begin{bmatrix}
S + 2 & 1 \\
-5 & S
\end{bmatrix}\]

\[B = \frac{1}{S^2 + 2S + 5} \begin{bmatrix}
S + 2 & -2 & -(S + 1) \\
-5 & -2S & 5 + 3S
\end{bmatrix}\]

\[H(s) = C(SI-A)^{-1}B + D = \frac{1}{S^2 + 2S + 5} \begin{bmatrix}
S^2 + 2S + 20 & -S^2 + 2S - 15 + 9S \\
2S + 24 & 8S - 4 - 14S - 18
\end{bmatrix}\]

\[= \begin{bmatrix}
H_{11}(s) & H_{12}(s) & H_{13}(s) \\
H_{21}(s) & H_{22}(s) & H_{23}(s)
\end{bmatrix}\]
Alternatively, the transfer function can be found from the state diagram using Mason's gain formula or block diagram reduction.

\[ \Delta = 1 + \frac{2}{3} + \frac{5}{s^2} = \frac{s^2 + 2s + 5}{s^2} \]

\[ N_{ii}(s) = \left. \frac{Y_1}{U_1} \right|_{U_2=0, U_3=0} = \frac{M_1\Delta_1 + M_2\Delta_2}{\Delta} \]
\[ = \frac{15s+1 \left( \frac{s^2 + 2s + 5}{s^2} \right)}{s^2 + 2s + 5} = \frac{s^2 + 2s + 20}{s^2 + 2s + 5} \]

\[ H_{12}(s) = \left. \frac{Y_1}{U_2} \right|_{U_1=0, U_3=0} = \frac{6s(1) + (-2) \left( \frac{s^2 + 2s + 5}{s^2} \right)}{s^2 + 2s + 5} \]
\[ = \frac{-2s + 2s - 10}{s^2 + 2s + 5} \]

Similarly, we can find the other transfer functions.
State-Transition Matrix

- The state-transition matrix is defined as a matrix that satisfies the linear homogeneous state equation
  \[ \frac{dx(t)}{dt} = Ax(t) \]

- Let \( \Phi(t) \) be the \( n \times n \) matrix that represents the state transition matrix; then it must satisfy the equation
  \[ \frac{d\Phi(t)}{dt} = A\Phi(t) \]

- Furthermore, let \( x(0) \) denote the initial state at \( t=0 \); then \( \Phi(t) \) is also defined by the matrix equation
  \[ x(t) = \Phi(t)x(0) \]

- One way of determining \( \Phi(t) \) is by taking the Laplace transform of both sides of linear homogeneous state equation
  \[ SX(s) - x(0) = AX(s) \]
  or \( X(s) = (sI-A)^{-1}x(0) \)

- Taking inverse L.T. of both sides, we get
  \[ x(t) = \mathcal{L}^{-1}[(sI-A)^{-1}]x(0) , \ t \geq 0 \]

- By comparing, we get
  \[ \Phi(t) = \mathcal{L}^{-1}[(sI-A)^{-1}] \]
• An alternative way of solving the homogeneous state equation is to assume the solution as in the classical method of solving differential equations as

$$X(t) = e^{At} x(0), \quad t \geq 0$$

where

$$e^{At} = I + At + \frac{1}{2!} A^2 t^2 + \frac{1}{3!} A^3 t^3 + \ldots$$

• It is easy to show that

$$\frac{d e^{At}}{dt} = Ae^{At}$$

Thus, we have

$$\Phi(t) = e^{At} = I + At + \frac{1}{2!} A^2 t^2 + \frac{1}{3!} A^3 t^3 + \ldots$$

**Significance of the State-Transition Matrix**

• Since the state-transition matrix satisfies the homogeneous state equation, it represents the free response of the system, i.e., it governs the response that is excited by the initial conditions only.

• The state transition matrix is dependent only upon the matrix $A$, and, therefore, it sometimes referred to as the state-transition matrix of $A$.

• As the name implies, the state-transition matrix completely defines the transition of the states from the initial time $t=0$ to any time $t$ when the inputs are zero.
Properties of the State-Transition Matrix

1. \( \Phi(0) = I \)

2. \( \Phi'(t) = \Phi(-t) \)

   Proof: \( \Phi(t) e^{At} = e^{At} e^{At} = I \)
   
   \[ e^{At} = \Phi(t) \]
   
   \( \Phi(t) = \Phi'(-t) \)

   - An interesting result from this property of \( \Phi(t) \) is that
     \[ x(0) = \Phi(-t)x(t) \]

     which means that the state-transition process can be considered as bilateral in time. That is, the transition in time can take place in either direction.

3. \( \Phi(t_2-t_1) \Phi(t_1-t_0) = \Phi(t_2-t_0) \), for any \( t_0, t_1, t_2 \)

   Proof \( \Phi(t_2-t_1) \Phi(t_1-t_0) = e^{A(t_2-t_1)} e^{(t_1-t_0)} e^{A(t_2-t_0)} = \Phi(t_2-t_0) \)

   - This property implies that a state-transition process can be divided into a number of sequential transitions

4. \( (\Phi(t))^k = \Phi(kt) \), for \( k \) = positive integers
State-Transition Equation

From the previous discussion, we have

\[ x(s) = (sI - A)^{-1} x(0^-) + (sI - A)^{-1} B u(s) \]

- The state-transition equation is obtained by taking the inverse Laplace transform of both sides:

\[ x(t) = \phi(t) x(0^-) + \int_0^t \phi(t - \tau) B u(\tau) d\tau \]

- The state-transition equation is useful only when the initial time is zero.

- In the study of control systems, especially discrete-data control systems, it is often desirable to break up a state-transition process into a sequence of transitions, so a more flexible initial time, for example, to may be chosen. Assume that the input \( u(t) \) is applied at \( t \geq 0 \).

- Let us start by setting \( t = t_0 \) and solving for \( x(t) \):

\[ x(t_0) = \phi(t_0) x(0) + \int_0^{t_0} \phi(t_0 - \tau) B u(\tau) d\tau \]

\[ \therefore x(0) = \phi(-t_0) x(t_0) - \phi(-t_0) \int_0^{t_0} \phi(t_0 - \tau) B u(\tau) d\tau \]

Substituting \( x(0) \) in \( x(t) \), we get

\[ x(t) = \phi(t - t_0) x(t_0) - \int_0^t \phi(t - \tau) B u(\tau) d\tau + \int_0^t \phi(t - \tau) B u(\tau) d\tau \]
\[ x(t) = \phi(t-t_0) x(t_0) + \int_{t_0}^{t} \phi(t-t) B U(\tau) d\tau \]

- Once the state equation is determined, the output vector can be expressed as a function of the initial state and the input vector simply by substituting \( x(t) \)

\[ y = C \phi(t-t_0) x(t_0) + \int_{0}^{t} C \phi(t-\tau) B U(\tau) d\tau + D U(t) \]

**Example 5.** Consider the state equation

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
-2 & -3
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} + \begin{bmatrix}
0 \\
1
\end{bmatrix} u
\]

The problem is to determine the state-transition matrix \( \phi(t) \) and the state vector \( x(t) \) for \( t \geq 0 \) when the input \( u = 1 \) for \( t \geq 0 \).

**Solution:**

\[
(S I - A)^{-1} = \begin{bmatrix}
s & -1 \\
2 & s+3
\end{bmatrix}^{-1} = \frac{1}{s^2 + 3s + 2} \begin{bmatrix}
s+3 & 1 \\
2 & s+1
\end{bmatrix}
\]

\[
\phi(t) = e^{(SI-A)t} = \frac{1}{s^2 + 3s + 2} \begin{bmatrix}
\frac{2}{s+1} & -1 \\
1 & -\frac{1}{s+2}
\end{bmatrix} \begin{bmatrix}
\frac{2}{s+1} & 2 \\
-\frac{1}{s+2} & \frac{2}{s+1} + \frac{1}{s+2}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
e^{-t} & -e^{-2t} & e^t & e^{-2t} \\
-2e^{-t} + 2e^{-2t} & -e^t & -2e^{-2t}
\end{bmatrix}
\]
Thus, we have
\[
L^2 \left( \frac{1}{s+3} - \frac{1}{s+1} \right) \left( \begin{array}{c} 1 \\ 1 \\ 0 \end{array} \right) = L \left( \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right).
\]

As an alternative, the second term of the state-transition equation can be obtained by

\[
x(t) = \left[ \begin{array}{c} 2e^{-t} - e^{-2t} \\ -2e^{-t} + e^{-2t} \\ 0 \end{array} \right] + \int_{t_0}^{t} \left[ \begin{array}{c} 2e^{-s} - e^{-2s} \\ -2e^{-s} + e^{-2s} \\ 0 \end{array} \right] ds.
\]