Numerical Approximations

- Several numerical approximation techniques will be discussed: some for differentiation and some for integration.

**Backward difference**

The backward difference is a simple technique that replaces the derivative of a function by

\[
\frac{dy(t)}{dt} \approx \frac{y(t)-y(t-T)}{T}
\]

![Graph of backward difference approximation]

Figure 4.5. Backward difference approximation.

- In the Laplace domain

\[
SY(s) = \frac{Y(s) - e^{sT}Y(s) + y(0+)}{T}
\]

- If \( y(0+) \) is small, then

\[
S \approx \frac{1 - e^{sT}}{T}
\]

\[
S \approx \frac{1 - z^{-1}}{T}
\]

Hence \[D(z) = G_c(s) \mid s = \frac{1 - z^{-1}}{T}\]
Example 4.1

Find the discrete approximation for

\[ G_c(s) = \frac{S}{S+a} = \frac{Y(s)}{X(s)} \]

\[ Y(s) = G_c(s) \cdot X(s) \]

\[ s \cdot Y(s) + a \cdot Y(s) = s \cdot X(s) \]

\[ a \cdot \frac{d}{dt} y(t) + a \cdot y(t) = \frac{d}{dt} x(t) \]

Now let

\[ \frac{d}{dt} y(t) = \frac{y(t) - y(t-T)}{T} \]

\[ \frac{d}{dt} x(t) = \frac{x(t) - x(t-T)}{T} \]

\[ \therefore \frac{y(t) - y(t-T)}{T} + a \cdot y(t) = \frac{x(t) - x(t-T)}{T} \]

Evaluating at \( t = nT \) yields

\[ y(nT) \left[ 1 + aT \right] = x(nT) - x(nT - T) + y(nT - T) \]

\[ \therefore y(nT) = \frac{1}{1 + aT} \left[ x(nT) - x(nT - T) + y(nT - T) \right] \]

\[ \therefore D(z) = \frac{1}{1 + aT} \frac{1 - z^{-1}}{1 - \frac{1}{1 + aT} z^{-1}} \]
Now using the relationship

\[ D(z) = G_c(s) \bigg|_{s = \frac{1 - z^{-1}}{T}} \]

\[ = \frac{s}{s + a} \bigg|_{s = \frac{1 - z^{-1}}{T}} \]

\[ = \frac{(1 - z^{-1})/T}{a + \frac{1 - z^{-1}}{T}} \]

\[ = \frac{1 - z^{-1}}{aT + 1 - z^{-1}} \]

\[ = \frac{1}{1 + aT} \frac{1 - z^{-1}}{1 - \frac{1}{1 + aT} z^{-1}} \]

**Forward Difference**

A similar numerical technique approximates

\[ \frac{d}{dt} y(t) \approx \frac{y(t + T) - y(t)}{T} \]

\[ y(t + T) - y(t) \]

\[ nt \quad nt+T \]

**Figure 4.6. Forward difference approximation**
In the Laplace domain
\[ S \cdot Y(s) = \frac{e^{sT}Y(s) - Y(s)}{s} + y(0+) \]
and, if \( y(0+) \) is neglected
\[
S \approx \frac{e^{sT}}{s} \approx \frac{e^{T}}{s} \]

Hence
\[
D(z) = G_c(s) \bigg|_{s = \frac{z^{-1}}{T}}
\]

**Example 4.2:**
Find a discrete version of \( G_c(s) \) using the forward difference.

\[ G_c(s) = \frac{s}{s+a} \]

\[
D(z) = \frac{s}{s+a} \bigg|_{s = \frac{z^{-1}}{T}} = \frac{(z^{-1})/T}{(z^{-1})/T + a} = \frac{z^{-1}}{z + (aT-1)}
\]

\[
= \frac{1 - z^{-1}}{1 + (aT-1)z^{-1}}
\]
• Rectangular rule

Suppose now that we try some numerical approximations to integrals and compare results. The idea there is to represent $G_c(s)$ as

$$G_c(s) = \frac{a_0 + a_1 s^{-1} + a_2 s^{-2} + \cdots + a_n s^{-n}}{1 + b_1 s^{-1} + b_2 s^{-2} + \cdots + b_n s^{-n}}$$

Each $s^{-1}$ represents an integrator in the s-domain. Hence, we can replace each integrator by its digital equivalent

$$s^{-1} = f(z)$$

or

$$s = \frac{1}{f(z)} = g(z)$$

A digital equivalent of $G_c(s)$ will be produced

**Left Side Rule**

![Figure 4.2](image.png)

**Left side rectangular rule**

$$y(t) = \int_{t}^{\infty} x(t) dt$$
Assume that the upper limit of the integral is $t = nT$. Hence

$$y(nT) = \int_0^{nT} x(t) \, dt$$

$$= T \sum_{\tau = 0}^{n-1} x(\tau T)$$

$$= y(nT) + T x(nT)$$

Taking $z$-transform of both sides, we get

$$Z \cdot Y(z) = Y(z) + T \cdot X(z)$$

$$\Rightarrow \frac{Y(z)}{X(z)} = D(z) = \frac{T z^{-1}}{1 - z^{-1}} = \frac{T}{z-1}$$

Hence, we have approximated the integration transfer function

$$\frac{1}{S} \approx \frac{T}{z-1} = f(z)$$

which gives the same result as the forward difference.
Right-Side Rule

\[ y(nT) = T \sum_{i=1}^{n} x(it) \]

\[ y((n+1)T) = T \sum_{i=1}^{n+1} x(it) = T \sum_{i=1}^{n} x(it) + TX((n+1)T) \]

\[ = y(nT) + T x((n+1)T) \]

Letting \( n = n-1 \)

\[ y(nT) = y((n-1)T) + T x(nT) \]

Hence, the transfer function is

\[ D(z) = \frac{T}{1-z^{-1}} \]

Consequently, we have approximated the integral

\[ \frac{1}{s} \approx \frac{T}{1-z^{-1}} = f(z) \]

Which yields the identical result as the backward difference
Trapezoidal Rule

The trapezoidal rule takes the average of the left and right-sides of the rectangles as shown in Figure 4.8.

\[
\text{Figure 4.9: Area approximated by the trapezoidal integration method.}
\]

Hence,

\[
y(nT) = \frac{T}{2} \sum_{k=0}^{n-1} \left[ x(2kT) + x(2(k+1)T) \right]
\]

\[
= \frac{1}{2} \left[ T \sum_{k=0}^{n-1} x(2kT) + T \sum_{k=1}^{n} x(2kT) \right]
\]

\[
\frac{1}{S} \cdot \mathcal{D}(z) \approx \frac{1}{2} \left[ \frac{Tz}{1-z^{-1}} + \frac{T}{1-z^{-1}} \right]
\]

\[
= \frac{T}{2} \frac{1+z^{-1}}{1-z^{-1}}
\]

Thus, we have approximated

\[
\frac{1}{S} \approx \frac{T}{2} \frac{1+z^{-1}}{1-z^{-1}} = f(z)
\]

This approximation is the familiar bilinear Z-transform.