Example 2:

Using the Bode diagram approach in the $W$-plane, design a digital controller for the system shown in Figure 1. The design specifications are

(i) Phase margin $> 50^\circ$
(ii) Gain margin $> 10$ dB
(iii) Static velocity error constant $k_v$ be 20 sec

The sampling period be 0.1 sec, or $T = 0.1$ sec. After the controller is designed, calculate the number of samples per cycle of damped sinusoidal oscillation

\[ \Lambda(t) \to \text{ } \boxed{D(z)} \to \frac{z}{q^i} \frac{K}{s(s+0.5)} \to C(t) \]

Figure 1. Digital Control System for example 2.
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Specs:
\[ K_v = 20 \]
\[ \text{PM} \geq 50^\circ \]
\[ \text{GM} > 10 \text{ dB} \]

Set:
\[ G(z) = (1-z^{-1})^3 \left[ \frac{K}{s^2(s+0.5)} \right] \]
\[ = 0.004918K \frac{z+0.9835}{(z-1)(z-0.9512)} \]
\[ K_v = \frac{1}{T} \lim_{z \to 1} (z-1)G(z) \]
\[ = 10 \left( 0.00498K \right) \frac{1.9835}{0.0488} = 20 \]
\[ \therefore K = 10 \]

For \( K = 10 \),
\[ G(w) = \frac{0.002(20-w)(w+2404.2)}{w(w+0.5002)} \]

The Bode plot of the uncompensated system is shown in Figure 1. It can be seen from this figure that
\[ \text{GM} = 0.077 \text{ dB} \]
\[ \text{PM} = 0.08^\circ \]

Therefore, we need a controller. The system is marginally stable, therefore, the leg controller would not work well.
Figure 1. Bode plot of the uncompensated system with $K = 10$. 

\[ \text{Bode Diagram} \]

Gm = 0.077139 dB (at 3.1785 rad/sec), Pm = 0.080128 deg (at 3.1621 rad/sec)
Since we need a phase of approximately $50^\circ$, lead controller by itself would not work well either. We either need a lead-lead controller or lead-lag controller. Let us design a lead-lag controller. We will first design a phase lead controller for a phase angle of $30^\circ$.

$$D_p(w) = \frac{1+q_1 w}{1+T_1 w}, \quad q_1 > 0$$

$$q_1 = \frac{1+5w 30^\circ}{1-5w 30^\circ} = 3$$

$$-10 \log_{10} 3 = -4.77 \text{dB}$$

The value of $w_m$ where the magnitude of the Bode plot is $-4.77 \text{dB}$ is $4.2$ rad/s

$$\therefore \frac{1}{T_1} = \sqrt{2} w_m = 7.27 \text{ rad/s}$$

$$\therefore D_p(w) = 3 \frac{w+2.43}{w+7.27}$$

The Bode plot of the system compensated by a phase lead controller is shown in Figure 2. It can be seen from this figure that the GM is $10.25 \text{ dB}$ and the PM is $25.1^\circ$. Now, we need to design a phase lag part of the controller to get an addition phase of $25^\circ$. To get this phase, we need to design another phase lead controller. Let $\phi_m = 90^\circ$. 

Bode Diagram

Gm = 10.744 dB (at 9.7333 rad/sec), Pm = 25.047 deg (at 4.1887 rad/sec)

Figure 2. Bode plot of the system with a phase-lead controller.
\[ B.4.15 \text{ (Cont'd)} \]

\[ D_2(W) = \frac{1 + a_2 T_2 W}{1 + T_2 W}, \quad a_2 > 0 \]

\[ Q_2 = 4.8 \]
\[-10 \log_{10} 3 = -6.92 \text{ dB} \]
\[ \frac{1}{T_2} = \sqrt{\alpha} \omega_m \]
\[ \omega_m^2 = 15 \]
\[ \therefore D_2(W) = 4.6 \frac{W + 3.26}{W + 15} \]

\[ \therefore D(W) = 13.8 \frac{(W + 2.42)(W + 3.26)}{(W + 7.27)(W + 15)} \]

\[ \therefore D(z) = 7.54 \frac{(z - 7.841)(z - 7.197)}{(z - 4.68)(z - 1.146)} \]

\[ \therefore D(z) G(z) = \frac{370.8 z^2 - 1929 z - 3392 z + 2058}{2 z^4 - 2.561 z^3 + 2.207 z^2 - 71 z + 63} \]

\[ \therefore C(z) \]
\[ \frac{R(z)}{R(z)} = \frac{370.8 z^3 - 1929 z^2 - 3392 z + 2058}{2 z^4 - 2.19 z^3 + 2.014 z^2 - 1.049 z + 2.92} \]

The step response of the system is shown in Figure 4.
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Bode Diagram

Gm = 8.1664 dB (at 19.212 rad/sec), Pm = 51.242 deg (at 7.1586 rad/sec)

Figure 3. Bode plot of the system with a lead-lead controller.
Figure 4. Step response of the compensated system.