Example 3:

Design a phase-lag controller in the \( \mathcal{W} \)-domain for the system shown in Figure 1. The design specifications are:

1. Phase margin \( > 50^\circ \)
2. Gain margin \( > 12 \text{ dB} \)
3. Static velocity error constant \( K_v \) with \( \varepsilon > 5 \text{ sec}^{-1} \)

![Diagram of the control system with Block Diagram Elements]

**Figure 1.** Digital control system for example 3.

The sampling period is \( T = 0.1 \text{ sec} \). After the controller is designed, make sure to check the time-domain specifications.
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\[ G(s) = \frac{5}{(s+1)(s+2)} \]

**Specs:**
- \( K_v = 5 \)
- \( PM = 50 \)^\circ
- \( GM = 12 \) dB

**Sol:**
\[ G(z) = (1-z^{-1})3 \left[ \frac{5}{(z+1)(z+2)} \right] \]
\[ = 0.02263 \frac{z + 0.906}{(z-0.9048)(z-0.8187)} \]

This is a type-0 system. However, to satisfy the \( K_v \) spec, we need a type-1 system. Therefore, we need a controller that includes an integrator. If the controller does not include an integrator, then

We will satisfy the \( K_v \) spec later on by adding a phase-lag controller. Let us first design a proportion plus compensator to satisfy the \( PM \) spec.
\[ D_1(z) = K \frac{z + z_1}{(z-1)} \]

Let us cancel the pole at 0.9048 with the zero of the PI controller.
\[ D_1(z) = K \frac{z - 0.9048}{(z-1)} \]
\[ G(z)D(z) = 0.02263K \frac{z+0.906}{(z-1)(z-0.8187)} \]

\[ G(w)D_i(w) = 0.006K \frac{(20-w)(w+406)}{w(w+1.9937)} \]

The Bode plot of the system with a PI compensator is shown in Figure 1. It can be seen that the PM = 43.4° and GM = 19 dB. However, we need a PM of 50°. Let us select K so that the PM = 50°. We need to decrease the magnitude by 2.67 dB. Let us redesign for a magnitude of 8 dB. (i.e., the entire magnitude curve must be lowered by 8 dB.) Thus, we require that the gain be set such that

\[ 20 \log K \text{ dB} = 8 \text{ dB} \]

\[ K = 0.4 \]

The Bode plot of the system with K = 0.4 is shown in Figure 2. It can be seen from this figure that the PM = 64.° and GM = 26.9 dB. Hence the specs are met.

\[ D(z) = 0.0091 \frac{z-0.9048}{(z-1)} \]

\[ G(z)D_i(z) = 0.0091 \frac{z+0.906}{(z-1)(z-0.8187)} \]

\[ K_V = \frac{1}{0.9} \lim_{z \to 0.1} \frac{0.0091(z+0.906)}{(z-1)(z-0.8187)} \]

\[ \therefore K_V = 0.9567 \]
Figure 1. Bode plot of the uncompensated system.
Bode Diagram

Gm = 28.89 dB (at 6.493 rad/sec), Pm = 63.986 deg (at 0.872 rad/sec)

Figure 2. Bode plot of the system compensated by a PI controller.
However, we need a $K_v$ of 5. Let us add a phase-lead controller to provide a gain $\frac{5}{0.9567} = 5.226$

Let $D_2(z) = \frac{z-z_1}{z-p_1}$

$\frac{1-z_1}{1-p_1} = 5.226$

Let $p_1 = 0.99$

$\therefore z_1 = 0.9477$

$\therefore D_2(z) = \frac{z-0.9477}{z-0.99}$

$D(z) = 0.4 \frac{z-0.9048}{z-1} \frac{z-0.9477}{z-0.99}$

$D(z)G(z) = 0.0091 \frac{(z+0.9061)(z-0.9477)}{(z-1)(z-0.8187)(z-0.99)}$

$\therefore K_v = \frac{1}{T} \lim_{z \to 1} D(z)G(z) = 10(0.0091)(1.9061)(0.0523) \frac{1}{1.1813(0.01)} = 5$

$\therefore \frac{C(z)}{R(z)} = \frac{0.0091z^2}{z^3 - 2.7996z^2 + 2.6188z - 0.8183}$

The step response of the system is shown in Figure 3.
Figure 3. Step response of the compensated system.