Example: In the circuit below, the switch is in the closed position for a long time before $t=0$ when it is opened instantaneously. Find the inductor current $y(t)$ for $t \geq 0$.

When the switch is in the closed position (for a long time), the inductor current is 2 amp and the capacitor voltage is 10V.

Solution: When the switch is open, the circuit becomes

\[
\begin{align*}
\frac{dy}{dt} + 2y(t) + 5\int_{-\infty}^{t} y(\tau) d\tau &= 10u(t) \\
\text{If} \quad y(t) &\leftrightarrow Y(s) \\
\frac{dy(t)}{dt} &\leftrightarrow sY(s) - y(0^-) \\
\text{and} \quad \int_{-\infty}^{t} y(\tau) d\tau &\leftrightarrow \frac{Y(s)}{s} + \frac{1}{5} \int_{0}^{\infty} y(\tau) d\tau
\end{align*}
\]
Now, we know

\[ u_c(t) = \frac{1}{c} \int_{-\infty}^{t} y(\tau) \, d\tau \]

\[ u_c(0) = \frac{1}{\sqrt{s}} \int_{-\infty}^{0} y(\tau) \, d\tau \]

\[ \int_{-\infty}^{0} y(\tau) \, d\tau = \frac{1}{\sqrt{s}} (10) = 2 \]

\[ \int_{-\infty}^{t} y(\tau) \, d\tau \iff \frac{y(s)}{s} + \frac{2}{s} \]

Taking the Laplace transform of both sides of differential equation, we get

\[ (sY(s) - 2) + 2Y(s) + 5 \left[ \frac{y(0)}{s} + \frac{2}{s} \right] = \frac{10}{s} \]

\[ (s + 2 + \frac{5}{s}) Y(s) = 2 \]

\[ Y(s) = \frac{2s}{s^2 + 2s + 5} \]

\[ = \frac{2(s+1) - 2}{(s+1)^2 + 2^2} \]

\[ Y(t) = \mathbf{e}^{-t} \left( 2 \cos t - \sqrt{2} \sin t \right) u(t) \]

\[ = \sqrt{2} \mathbf{e}^{-t} \cos (2t + 26.6^\circ) u(t) \]
Example: (a) Find the transfer function of the system represented by the differential equation

\[ \frac{d^2 y}{dt^2} + 5 \frac{dy}{dt} + 6 y(t) = \frac{df(t)}{dt} + f(t) \]

\[ \therefore (s^2 + 5s + 6) Y(s) = (s + 1) F(s) \]

\[ \therefore H(s) = \frac{Y(s)}{F(s)} = \frac{s + 1}{(s + 2)(s + 3)} \]

(b) Find \( y(t) \) if \( f(t) = 3 e^{-st} u(t) \) and all initial conditions are zero.

\[ Y(s) = H(s) F(s) \]

\[ F(s) = \frac{3}{s + 5} \]

\[ \therefore Y(s) = \frac{3(s + 1)}{(s + 2)(s + 3)(s + 5)} \]

\[ = \frac{-2}{s + 5} - \frac{1}{s + 2} + \frac{3}{s + 3} \]

\[ \therefore y(t) = (-2 e^{-5t} - e^{-2t} + 3 e^{-3t}) u(t) \]
Example:

For an LTI C system with the transfer function

\[ H(s) = \frac{s + 5}{s^2 + 4s + 3} \]

(a) Describe the differential equation relating the input \( f(t) \) and output \( y(t) \).

(b) Find the system response \( y(t) \) to the input \( f(t) = e^{-2t}u(t) \) if the system is initially in zero state.

\[ H(s) = \frac{Y(s)}{F(s)} = \frac{s + 5}{s^2 + 4s + 3} \]

\[ S^2 Y(s) + 4S Y(s) + 3Y(s) = SF(s) + SF(s) \]

Taking the inverse transform of both sides, we get

\[ \frac{d^2 y(t)}{dt^2} + 4 \frac{dy(t)}{dt} + 3y(t) = \frac{df(t)}{dt} + 5f(t) \]

\[ f(t) = e^{-2t}u(t) \]

\[ \therefore F(s) = \frac{1}{s + 2} \]

\[ \therefore Y(s) = \frac{s + 5}{(s + 2)(s + 1)(s + 3)} \]

\[ = \frac{2}{s + 1} - \frac{3}{s + 2} + \frac{1}{s + 3} \]

\[ \therefore y(t) = \left[ 2e^{-t} - 3e^{-2t} + 3e^{-3t} \right] u(t) \]
An Example of Electrical Circuit

\[ 20V \quad + \quad \begin{array}{c}
\int_{0}^{t} y_1(z) \, dz \\
\int_{-\infty}^{0} y_1(z) \, dz
\end{array} \]

\[ \frac{1}{10} \int_{0}^{t} y_1(z) \, dz + \frac{1}{5} (y_1 - y_2) = 20 \, \text{V(t)} \quad (1) \]

\[ y_2 + \frac{1}{2} \frac{d}{dt} y_2 + \frac{1}{5} (y_2 - y_1) = 0 \quad (2) \]

\[ u_c(t) = \frac{1}{C} \int_{-\infty}^{t} y_1(z) \, dz \]

\[ -\int_{-\infty}^{0} y_1(z) \, dz = u_c(0-) = 5 \]

Taking L.T. of equation (1), we get

\[ \frac{Y_1(s)}{S} + \frac{5}{S} + \frac{1}{5} (Y_1(s) - Y_2(s)) = \frac{20}{S} \]

or

\[ 5Y_1(s) + 25 + S(Y_1(s) - Y_2(s)) = 100 \]

\[ (S+5)Y_1(s) - SY_2(s) = 75 \quad (3) \]

Taking L.T. of equation (2), we get

\[ Y_2(s) + \frac{1}{2} [S Y_2(s) - 4] + \frac{1}{5} (Y_2(s) - Y_1(s)) = 0 \]

\[ -\frac{1}{5} Y_1(s) + \left( \frac{1}{2} S + \frac{6}{5} \right) Y_2(s) = 0 \]

or

\[ -2Y_1(s) + (5S + 12) Y_2(s) = 20 \quad (4) \]
Writing equations 3 and 4 in the matrix form, we get

\[
\begin{bmatrix}
  s+5 & -s \\
  -2 & 5s+12
\end{bmatrix}
\begin{bmatrix}
  y_1(s) \\
  y_2(s)
\end{bmatrix}
= 
\begin{bmatrix}
  75 \\
  20
\end{bmatrix}
\]

Using Cramer's rule, we get

\[
y_1(s) = \frac{\begin{vmatrix}
  75 & -s \\
  20 & 5s+12
\end{vmatrix}}{\begin{vmatrix}
  s+5 & -s \\
  -2 & 5s+12
\end{vmatrix}} = \frac{75(5s+12)+20s}{(s+5)(5s+12)-2s}
\]

\[
= \frac{395s + 900}{5s^2 + 35s + 60} = \frac{79s + 180}{s^2 + 7s + 12} = \frac{-57 + 136 e^{-4t}}{s+3} e^{-3t}
\]

\[\hat{y}_1(t) = (-57 e^{-3t} + 136 e^{-4t})u(t)\]

Similarly,

\[
y_2(s) = \frac{2(2s+25)}{s^2 + 7s + 12} = \frac{38}{s+3} - \frac{34}{s+4}
\]

\[\hat{y}_2(t) = (38 e^{-3t} - 34 e^{-4t})u(t)\]
Example: Find the impulse response $h(t)$ of the system described by

\[(D^2 + 3D + 2) y(t) = D f(t)\]

**Sol:**

\[
\left(\frac{s^2 + 3s + 2}{s^2 + 3s + 2}\right) y(s) = s F(s)
\]

\[\therefore \frac{y(s)}{F(s)} = H(s) = \frac{s}{s^2 + 3s + 2}\]

If $f(t) = s(t)$

then $F(s) = 1$

\[\therefore \quad y(s) = H(s) = \frac{s}{s^2 + 3s + 2}\]

Therefore,

\[y(t) = \mathcal{L}^{-1}[H(s)] = h(t)\]

\[= \frac{s}{(s+1)(s+2)}\]

\[= \frac{-1}{s+1} + \frac{2}{s+2}\]

so \[h(t) = (-e^{-t} + 2e^{-2t}) u(t)\]

**Remember** \[h(t) = \mathcal{L}^{-1}[H(s)]\]
Analysis of Electrical Networks: The Transformed Network Method

- It is possible to analyze electrical networks without having to write the integro-differential equations.
- This requires representing a network in the "frequency domain" where all the voltages and currents are represented by their Laplace Transform.

Assume Zero Initial Conditions

\[ f(t) \quad \rightarrow \quad y(t) \quad \rightarrow \quad \frac{1}{C} \int y(t) \, dt \]

\[ V_R(t) = R \cdot y(t) \quad \Rightarrow \quad V_R(s) = R \cdot Y(s) \]
\[ V_L(t) = L \frac{dy}{dt} \quad \Rightarrow \quad V_L(s) = SL \cdot Y(s) \]
\[ V_C(t) = \frac{1}{C} \int y(t) \, dt \quad \Rightarrow \quad V_C(s) = \frac{1}{Cs} \cdot Y(s) \]

Since \[ V_R(t) + V_L(t) + V_C(t) = f(t) \]
Taking the L.T. of both sides, we get
\[ V_R(s) + V_L(s) + V_C(s) = F(s) \]
\[ (R + SL + \frac{1}{Cs}) Y(s) = F(s) \]

Therefore, we can redraw the circuit in the frequency domain as

\[ \text{\includegraphics[width=0.5\textwidth]{circuit}} \]

- Thus, in the "frequency domain," the voltage-current relationship of an inductor and a capacitor are algebraic. These elements behave like resistors of "resistances" \( Ls \) and \( \frac{1}{Cs} \), respectively. The "generalized resistance" of an element is called the impedance and is given by the ratio \( \frac{Y(s)}{E(s)} \) for the element (under zero initial conditions). Therefore,

\[
Z_R = R \\
Z_L = SL \\
\text{and } Z_C = \frac{1}{SC}
\]

Thus, from the circuit in the frequency domain, we have

\[ Y(s) = \frac{F(s)}{Z_T} = \frac{F(s)}{R+SL+\frac{1}{SC}} \]
Circuit With Initial Conditions

**Series Circuit**

\[ V_R(t) = R \cdot y(t) \quad \Rightarrow \quad V_R(s) = R \cdot Y(s) \]

\[ V_L = L \cdot \frac{d}{dt} y(t) \quad \Rightarrow \quad V_L(s) = L \left( s \cdot Y(s) - y(0^-) \right) = L(s \cdot Y(s) - V_L(0^-)) = sL \cdot Y(s) - L \cdot V_L(0^-) \]

\[ V_C = \frac{1}{C} \int_{-\infty}^{t} y(t) \, dt \quad \Rightarrow \quad \frac{1}{C} \left[ \frac{Y(s)}{s} + \frac{1}{s} \int_{-\infty}^{0^-} y(t) \, dt \right] = V_C(s) \]

\[ V_C(0^-) = \frac{1}{C} \int_{-\infty}^{0^-} y(t) \, dt \]

\[ \therefore V_C(s) = \frac{1}{Cs} \cdot Y(s) + \frac{V_C(0^-)}{s} \]

Therefore, the series circuit can be drawn in the frequency domain as

[Diagram of the circuit with frequency domain representation]
Parallel Circuit with zero Initial Conditions

\[ f(t) \uparrow \quad R \quad L \quad C \quad + \quad u(t) = y(t) \quad - \]

Redrawing the circuit in the frequency domain, we get

\[ F(s) \uparrow \quad R \quad \frac{1}{sL} \quad \frac{1}{sC} \quad + \quad y(s) \quad - \]

\[ F(s) \uparrow \quad Y(s) \quad \text{Admittance} \]

\[ Y_T = Y_R + Y_L + Y_C = \frac{1}{R} + \frac{1}{sL} + sC \]

\[ = \frac{SL + R + s^2RLC}{sRL} \]

\[ \therefore Z_T = \frac{sRL}{s^2RLC + SL + R} \]

\[ \therefore Y(s) = \frac{sRL}{s^2RLC + SL + R} F(s) \]
Parallel Circuit with Initial Conditions

\[ f(t) \quad \begin{array}{c}
\uparrow \\
\end{array} \quad \begin{array}{c}
R \\
\downarrow \\
L \\
C \\
\end{array} \quad z_R(t) = f(t) \quad z_L(t) = \frac{1}{L} \int y(t) dt \quad z_C(t) = C \frac{dy(t)}{dt} \]

\[ z_R(t) + z_L(t) + z_C(t) = f(t) \]

\[ i_R(s) = \frac{Y(s)}{R} \]
\[ i_L(s) = \frac{1}{L} \int y(t) dt \]
\[ i_C(s) = C \frac{dy(t)}{dt} \]

\[ \begin{align*}
\dot{v}_R(t) &= \frac{y(t)}{R} \\
\dot{v}_L(t) &= \frac{1}{L} \int y(t) dt \\
\dot{v}_C(t) &= C \frac{dy(t)}{dt}
\end{align*} \]

\[ v_C(0-) = \frac{1}{L} \int y(t) dt \]

\[ v_L(0-) = \frac{1}{L} \int y(t) dt \]

\[ v_R(0-) = \frac{y(0)}{R} \]

\[ \therefore i_L(s) = \frac{1}{sL} \gamma(s) + \frac{v_L(0-)}{s} \]

\[ i_C(s) = C \left[ s \gamma(s) - \gamma(0-) \right] \]

\[ = C \left[ s \gamma(s) - v_C(0-) \right] \]

\[ = \frac{1}{sC} \gamma(s) - C v_C(0-) \]

Therefore, the circuit can be drawn in the frequency domain as

\[ F(s) \quad \begin{array}{c}
\uparrow \\
\end{array} \quad \begin{array}{c}
R \\
\downarrow \\
L \\
C \\
\end{array} \quad \begin{array}{c}
\frac{1}{sC} \\
\uparrow \\
\end{array} \quad v_C(0-) \]
Summary:

- An inductor L with an initial current $I_L(0^-)$ can be represented in the frequency domain as

\[ V(t) \xrightarrow{\text{s-domain}} I(s) \]

- Similarly, a capacitor C with an initial voltage $V_C(0^-)$ can be represented in the frequency domain as

\[ V(t) \xrightarrow{\text{s-domain}} I(s) \]
An Example:

\[ +V_C(t) = 5V \quad 1A \]

\[
\begin{pmatrix}
20V \\
+1F \\
- \frac{1}{5} -2 \\
\frac{1}{5}H \\
\end{pmatrix}
\]

Find \( y_1(t) \) and \( y_2(t) \)

**SOL:** Transforming the circuit in the frequency domain, we get

\[
\begin{pmatrix}
\frac{20}{5} \\
\frac{1}{5} \text{ } \frac{5}{5} \\
\text{ } \text{ } \frac{1}{5} \\
\text{ } \text{ } \text{ } \frac{5}{2} \\
\end{pmatrix}
\]

\[
\frac{Y_1(s)}{5} + \frac{Y_1(s) - Y_2(s)}{5} = \frac{15}{5}
\]

\[
- \left( \frac{1}{5} \right) Y_1(s) - \frac{1}{5} Y_2(s) = \frac{15}{5}
\]

and

\[
- \frac{1}{5} Y_1(s) + \left( \frac{6}{5} + \frac{5}{2} \right) Y_2(s) = 2
\]

Solve \( Y_1(s) \) and \( Y_2(s) \) and then find \( y_1(t) \) and \( y_2(t) \)