Laplace Transform Solution of Differential and Integro-Differential Equations

Example: Solve the second-order differential equation
\[ \frac{d^2 y}{dt^2} + 5 \frac{dy}{dt} + 6 y(t) = \frac{df}{dt} + f(t) \]

if the initial conditions are \( y(0^-) = 2 \) and \( \dot{y}(0^-) = 1 \), and the input \( f(t) = e^{4t} u(t) \).

Sol:
\[ \frac{d^2 y}{dt^2} + 5 \frac{dy}{dt} + 6 y(t) = \frac{df}{dt} + f(t) \]

Let \( y(t) \leftrightarrow Y(s) \)

Then using the time differentiation property, we get
\[ [s^2 Y(s) - sy(0^-) - \dot{y}(0^-)] + 5 [sY(s) - y(0^-)] + 6 Y(s) = SF(s) - f(0^-) + F(s) \]

\[ (s^2 + 5s + 6)Y(s) - (2s + 11) = (s + 1) \frac{1}{s + 4} \]

\[ (s^2 + 5s + 6)Y(s) = (2s + 11) + \frac{s + 1}{s + 4} \]

\[ (s^2 + 5s + 6)Y(s) = \frac{2s^2 + 20s + 45}{s + 4} \]

\[ \therefore Y(s) = \frac{2s^2 + 20s + 45}{(s + 4)(s + 3)(s + 2)} \]
Expanding the right-hand side into partial fractions, we get

\[ Y(s) = \frac{13/2}{s+2} - \frac{3}{s+3} - \frac{3/2}{s+4} \]

\[ \therefore y(t) = \left(\frac{13}{2} e^{-2t} - 3 e^{-3t} - \frac{3}{2} e^{-4t}\right) u(t) \]

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**Zero-Input and Zero-State Components of Response**

In the previous example, we have

\[ Y(s) = \frac{2s+11}{s^2+5s+6} + \frac{s+1}{(s+4)(s^2+5s+6)} \]

Zero-input component:

\[ = \left[ \frac{1}{s+2} - \frac{s}{s+3} \right] \]

Zero-state component:

\[ = \left[ \frac{-1/2}{s+2} + \frac{2}{s+3} - \frac{3/2}{s+4} \right] \]

Take the inverse Laplace transform of this equation yields

\[ y(t) = \left(7 e^{-2t} - 5 e^{-3t}\right) u(t) + \left(-\frac{1}{2} e^{-2t} + 2 e^{-3t} - \frac{3}{2} e^{-4t}\right) u(t) \]

Zero-input response

\[ = \left(\frac{13}{2} e^{-2t} - 3 e^{-3t} - \frac{3}{2} e^{-4t}\right) u(t) \]

Zero-state response

Total response