The Inverse Laplace Transform

- The general formula for recovering $X(t)$ from $X(s)$ is the complex line integral

$$X(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} X(s)e^{st}ds$$

*Do Not Use This Formula*

- Rather, to perform the inverse Laplace transform, we will merely manipulate the given expression until we see patterns we recognize from the Laplace transform table.

- This is basically a heuristic scheme and is one that will become more obvious with practice.

- Note that knowing ROC is critical to performing the inverse Laplace transform as we will see later.
Example: Find the inverse Laplace transform of

\[ X(s) = \frac{1}{s+3} \]

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\[ X(t) = e^{-3t} u(t), \quad \text{ROC: } \text{Re}(s) > -3 \]

\[ X(t) = -e^{-3t} u(t), \quad \text{ROC: } \text{Re}(s) < -3 \]

\[ F(s) = \frac{1}{s+3} \]
More complex transforms can be evaluated by first splitting the transform into simpler expressions using the process of partial fraction expansion.

**Example:** Find the inverse Laplace transform of

\[ \mathcal{F}(s) = \frac{3s+5}{s^2+3s+2}, \quad \text{ROC: } -2 < \text{Re}(s) < -1 \]

**Sol:**

Splitting \( \mathcal{F}(s) \) using partial fraction expansion

\[ \mathcal{F}(s) = \frac{A}{s+2} + \frac{B}{s+1} \]

\[ A = \left. \frac{3s+5}{s+1} \right|_{s=-2} = \frac{-1}{-1} = 1 \]

\[ B = \left. \frac{3s+5}{s+2} \right|_{s=-1} = \frac{2}{1} = 2 \]

\[ \mathcal{F}(s) = \frac{1}{s+2} + \frac{2}{s+1} \]

This term must have ROC of \( \text{Re}(s) < -1 \)

\[ \therefore \mathcal{F}(t) = e^{-2t} u(t) - 2e^{-t} u(-t) \]
• Sometimes, however, factoring the denominator is not always the best thing to do.

• In the case of second-order polynomial with complex roots, completing the square is generally the best procedure.

**Example:**
Find the inverse Laplace transform of

\[ F(s) = \frac{1}{s^2 + 4s + 40} \]

**Roc:** \( \text{Re}(s) > -2 \)

**Sol:**

\[ F(s) = \frac{1}{s^2 + 4s + 40} \]

\[ = \frac{1}{s^2 + 4s + 4 + 36} \]

\[ = \frac{\frac{1}{6}}{(s+2)^2 + 6^2} \]

\[ = \frac{1}{6} \cdot \frac{6}{(s+2)^2 + 6^2} \]

\[ \therefore f(t) = \frac{1}{6} e^{-2t} \sin(6t) u(t) \]
Example: Find the inverse Laplace transform of

\[ F(s) = \frac{2s^2 + 5}{s^2 + 3s + 2} \]

Sol:

\[ \frac{2}{s + 3s + 2} \left( \frac{2s^2 + 5}{s^2 + 4 + 6s} \right) \]

\[ = -6s + 1 \]

\[ \therefore F(s) = 2 + \frac{-6s + 1}{s^2 + 3s + 2} = 2 + \frac{-6s + 1}{(s+1)(s+2)} \]

Let \[ \frac{-6s + 1}{s^2 + 3s + 2} = \frac{R_1}{s+1} + \frac{R_2}{s+2} \]

\[ \because R_1 = \left. \frac{-6s + 1}{s + 2} \right|_{s = -1} = \frac{7}{1} = 7 \]

\[ R_2 = \left. \frac{-6s + 1}{s + 1} \right|_{s = -2} = \frac{13}{-1} = -13 \]

\[ \therefore F(s) = 2 + \frac{7}{s+1} - \frac{13}{s+2} \]

\[ \therefore f(t) = 2 \delta(t) + (7e^{-t} - 13e^{-2t}) u(t) \]
Example: Find the inverse Laplace transform of

\[ F(s) = \frac{6(s+34)}{s(s^2+10s+34)} \]

Solution:

\[
\frac{6(s+34)}{s(s^2+10s+34)} = F(s) = \frac{A}{s} + \frac{Bx+C}{s^2+10s+34}
\]

\[
A = \frac{6(s+34)}{s^2+10s+34} \bigg|_{s=0} = \frac{6 \times 34}{34} = 6
\]

Clearing the fractions by multiplying both sides by \( s(s^2+10s+34) \),

\[
6(s+34) = 6(s^2+10s+34) + (Bx+C)s
\]

\[
= (6+B) s^2 + (60+C) s + 204
\]

Equating the coefficients of \( s^2 \) and \( s \) both sides, we get

\[
0 = 6+B \implies B = -6
\]

\[
6 = 60+C \implies C = -54
\]

\[ \therefore F(s) = \frac{6}{s} + \frac{-6s-54}{s^2+10s+25+9} \]

\[ = \frac{6}{s} - \frac{6(s+5) + 24}{(s+5)^2 + 3^2} \]

\[ = \frac{6}{s} - \frac{6(s+5)}{(s+5)^2 + 3^2} - 8 \frac{3}{(s+5)^2 + 3^2} \]
\[ f(t) = \left[ 6 - e^{5t} (6 \cos 3t + 8 \sin 3t) \right] u(t) \]

Now,
\[ A \cos bt + B \sin bt = C \cos(bt + \theta) \]
\[ = C \cos bt \cos \theta - C \sin bt \sin \theta \]

\[ A = C \cos \theta \]
\[ B = -C \sin \theta \]
\[ A^2 + B^2 = C^2 \]
\[ \alpha = \sqrt{A^2 + B^2} \]
\[ C = \sqrt{A^2 + B^2} \]
\[ \tan \theta = -\frac{B}{A} \]
\[ \theta = \alpha \tan^{-2} \left( -\frac{B}{A} \right) \]

\[ A = 6 \]
\[ B = 8 \]
\[ \therefore C = \sqrt{36 + 64} = 10 \]
\[ \theta = \alpha \tan^{-2} \left( -\frac{8}{6} \right) = -53.1^\circ \]

\[ f(t) = \left[ 6 - 10 e^{5t} \cos (3t - 53.1^\circ) \right] u(t) \]
\[ = \left[ 6 + 10 e^{5t} \cos (3t + 126.9^\circ) \right] u(t) \]
Another Method of finding the values of $B$ and $C$

\[
\frac{6(s+34)}{s(s^2+10s+34)} = \frac{6}{s} + \frac{Bs+C}{s^2+10s+34}
\]

Multiply both sides by $s$, we get

\[
\frac{6(s+34)}{s^2+10s+34} = \frac{6}{s} + \frac{Bs^2+C}{s^2+10s+34}
\]

\[
\frac{\frac{6}{s} + \frac{34}{s^2}}{1 + \frac{10}{s} + \frac{34}{s^2}} = \frac{6}{s} + \frac{B + \frac{6}{s}}{1 + \frac{10}{s} + \frac{34}{s^2}}
\]

Let $s \to \infty$, we get

\[0 = 6 + B\]

\[\therefore B = -6\]

\[
\frac{6(s+34)}{s(s^2+10s+34)} = \frac{6}{s} + \frac{-6s+C}{s^2+10s+34}
\]

To find $C$, we let $s$ take any convenient value, say $s=1$, we get

\[
\frac{6\times35}{45} = 6 + \frac{-6+C}{45}
\]

\[6+35 = 6\times45 + (C-6)\]

\[\therefore C = -54\]
Example: Find $f(t)$ if

$$F(s) = \frac{8s+10}{(s+1)(s+2)^2}$$

Step:

$$F(s) = \frac{A}{s+1} + \frac{B}{(s+2)^2} + \frac{C}{s+2}$$

$$A = \frac{8s+10}{(s+2)^2} \bigg|_{s=-1} = \frac{2}{1} = 2$$

$$B = \frac{8s+10}{s+1} \bigg|_{s=-2} = \frac{-6}{-1} = 6$$

$$C = \frac{1}{11} \frac{d}{ds} \left[ \frac{8s+10}{s+1} \right] \bigg|_{s=-2} = \frac{(s+1)(8)-(8s+10)}{(s+1)^2} \bigg|_{s=-2}$$

$$= \frac{-2}{(s+1)^2} \bigg|_{s=-2} = \frac{-2}{1} = -2$$

$$\therefore F(s) = 2 \frac{1}{s+1} + 6 \frac{1}{(s+2)^2} - 2 \frac{1}{s+2}$$

$$\therefore f(t) = \left[ 2e^{-t} + e^{2t}(6t-2) \right] u(t)$$
Alternate method: A hybrid of Heaviside and Short-cuts

- A and B can be easily found using the Heaviside method.
- To find C, we can do the following:

\[
\frac{8s+10}{(s+1)(s+2)^2} = \frac{2}{s+1} + \frac{6}{(s+1)^2} + \frac{c}{s+2}
\]

\[
8s+10 = 2(s+2)^2 + 6(s+1) + c(s+1)(s+2)
\]

Equating the coefficient of \(s^2\) terms on both sides, we get

\[
0 = 2 + c
\]

\[
\therefore c = -2
\]