The Laplace Transform

- The Laplace transform is the most important tool in the analysis of continuous-time system.
- It is used to decompose a signal into the sum of complex exponentials.
- Since the output of a system for an exponential input is easily determined, the Laplace transform can be used to determine the output for virtually any input.

Definition of Laplace transform

Bilateral Laplace transform

\[ X(s) = \int_{-\infty}^{\infty} x(t)e^{-st} \, dt \]

Inverse Laplace transform

\[ x(t) = \frac{1}{2\pi j} \int_{C} X(s)e^{st} \, ds \]

- The formula for the inverse transform says that a signal \( x(t) \) can be expressed as the sum of an infinite number of appropriately scaled exponentials of the form \( X(s)e^{st} \).
- For the inverse transform, the contour of integration is a straight line parallel to the \( j\omega \)-axis for any value of \( s \) in the transform's region of convergence.
- You may think it is hard. Don't worry there's an easier way to take the inverse transform as we will see later.

**Unilateral Vs Bilateral**

- Bilateral Laplace transform is used for signals that exist over the entire time interval from \(-\infty\) to \(\infty\). Bilateral Laplace transform is also called two-sided Laplace transform.

- Most people concerned with the "real world" (e.g., control system designers) almost always use the unilateral formula where the limit of integration is from 0 to \(\infty\).

**Region of Convergence (ROC)**

- ROC of a Laplace transform is defined as the set of values of \(s\) for which the Laplace transform integral can be evaluated (i.e., it does not blow up).

**Example:**

If \(X(t) = e^{2t}u(t)\), find \(X(s)\) and ROC.

**Sol:**

\[
X(s) = \int_{0}^{\infty} e^{2t} e^{-st} dt = \int_{0}^{\infty} e^{(2-s)t} dt
\]

\[
= \left. \frac{e^{(2-s)t}}{2-s} \right|_{0}^{\infty}
\]

For \(2-s \neq 0\), the integral converges for \(s < 2\).

ROC: \(s < 2\).
\[
\lim_{s \to \infty} \frac{(s+1)}{s^2} = \frac{1}{s} < 0
\]
\[
\lim_{s \to -2} \frac{1}{s-2} = \frac{1}{s-2} < 0
\]

**Pole:** A pole is defined as a value of \( s \) that causes the denominator of the transform to become zero.

- For the above example, \( s = 2 \) is the pole.

**Rules of the ROC**

- The ROC is always a region of the \( s \)-plane to the left or right of a vertical line, or a strip between two vertical lines.
- The ROC never contains any poles.
- If \( x(t) \) is right-sided, then the ROC is right-sided, i.e., \( \text{Re}(s) > a \), where \( a \) is the rightmost pole.

Right-sided signal
• If \( x(t) \) is left-sided, the ROC is left-sided, i.e., \( \text{Re}(s) < a \), where \( a \) is the \( \text{Re}(\text{leftmost pole}) \).

![Left-sided signal diagram]

• If \( x(t) \) is two-sided or the sum of a left- and right-sided signal, the ROC is either a strip \( (a < \text{Re}(s) < b) \), or else the individual ROC's will not overlap, producing the null set.

![Two-sided signal diagram]

• If \( x(t) \) is of finite duration, the ROC is the entire \( s \)-plane.

![Finite duration signal diagram]
Determination of Laplace Transform

Unit Step

\[ f(t) = u(t) \]

\[ F(s) = \int_{-\infty}^{\infty} f(t) e^{-st} dt \]

\[ = \int_{0}^{\infty} e^{-st} dt \]

\[ = \left. \frac{-e^{-st}}{s} \right|_{0}^{\infty} \]

\[ = \frac{1}{s}, \quad \text{Re}(s) > 0 \]