For the Linear Time-Invariant Continuous (LTI C) System shown above, we have

\[ \frac{d^2 y}{dt^2} + \frac{R}{L} \frac{dy}{dt} + \frac{1}{LC} y(t) = x(t) \]

Let \( R = 3 \Omega \), \( L = 1 \) H and \( C = \frac{1}{2} \) F and \( x(t) = u(t) \)

\[ \frac{d^2 y}{dt^2} + 3 \frac{dy}{dt} + 2 y(t) = u(t) \]

- We can solve the differential equation and find the total response by using superposition.

\[ \text{Total response} = \text{zero-input response} + \text{zero-state response} \]

- Zero-state response is the response due to input when the initial conditions are zero.
- Zero-input response is the response due to initial conditions when the input is zero.

**Zero-input response:** Let \( v_C(0) = 1 \) and \( i(0) = 1 \).

\[ \frac{d^2 y}{dt^2} + 3 \frac{dy}{dt} + 2 y = 0 \]

**Characteristic Polynomial:** \( \lambda^2 + 3\lambda + 2 = 0 \)

\( (\lambda + 1)(\lambda + 2) = 0 \)
Therefore, \( \lambda_1 = -1 \) and \( \lambda_2 = -2 \)

where \( \lambda_1 \) and \( \lambda_2 \) are the solutions of the characteristic polynomial.

Thus, the solution of the differential equation is

\[
y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} = c_1 e^{-t} + c_2 e^{-2t}, \quad t > 0
\]

Now, we can find \( c_1 \) and \( c_2 \) using the initial condition. First, we need to find \( y(0) \) and \( y'(0) \) from \( u_c(0) \) and \( t(0) \).

We know, \( y(0) = u_c(0) = 1 \) \( \text{ (since } y(t) = u_c(t) \) and

\[
y'(t) = c \frac{du_c}{dt} = c \frac{dy(t)}{dt}
\]

\[
\therefore \quad y'(0) = \frac{1}{2} \frac{dy(0)}{dt}
\]

\[
\Rightarrow \quad \frac{dy(0)}{dt} = 2 \quad y(0) = 2
\]

Since \( y(t) = c_1 e^t + c_2 e^{-2t} \) \( \Rightarrow \) \( y(0) = c_1 + c_2 = 1 \)

\[
\therefore \quad y(t) = -c_1 e^{-t} - 2c_2 e^{-2t} \Rightarrow y(0) = -c_1 - 2c_2 = 2
\]

\[
\therefore \quad c_1 + 2c_2 = -2
\]

Using Cramer's rule,

\[
c_1 = \begin{vmatrix} -2 & 2 \\ 1 & 2 \end{vmatrix} = \frac{2+2}{2-1} = 4
\]

\[
c_2 = \begin{vmatrix} 1 & -2 \\ 1 & 2 \end{vmatrix} = \frac{-2-1}{2-1} = -3
\]

\[
\therefore \quad y(t) = 4 e^{-t} - 3 e^{-2t}, \quad t > 0
\]

\[
= (4 e^t - 3 e^{2t}) y(t)
\]
(2) **Zero-State Response**, i.e. response due to input considering initial condition to be zero.

Since the input is a constant voltage of 1V, we can obtain the output which is the voltage across the capacitor by inspection in the steady-state. Since in the steady-state, the capacitor is an open-circuit and the input is a short-circuit, we have

\[ V(\infty) = 1V \]

**Total response**:

\[ y(t) = 4e^{-t} - 3e^{-2t} + 1, \quad t \geq 0 \]

\[ = (1 + 4e^{-t} - 3e^{-2t})u(t) \]
Example 2:

\[ M \frac{d^2 y}{dt^2} + B \frac{dy}{dt} + Ky = x \]

Let \( \frac{B}{M} = 5 \text{ s}^{-1} \), \( \frac{K}{M} = 4 \text{ s}^{-2} \) and \( x(t) = u(t) \)
\( y(0) = 3 \text{ cm}, \quad \dot{y}(0) = 0.0 \text{ cm s}^{-1} \)

**Zero-input response (response due to I.C's):**

\[ \frac{d^2 y}{dt^2} + 5 \frac{dy}{dt} + 4y = 0 \]

\( \lambda^2 + 5\lambda + 4 = 0 \)
\( (\lambda + 4)(\lambda + 1) = 0 \)

\( \therefore \ y(t) = c_1 e^{-t} + c_2 e^{-4t} \quad \forall t > 0 \)
\( \dot{y}(t) = -c_1 e^{-t} - 4c_2 e^{-4t} \)

\( y(0) = c_1 + c_2 = 3 \)
\( \dot{y}(0) = -c_1 - 4c_2 = 0 \implies c_1 = -4c_2 \)
\( \therefore -4c_2 + c_2 = 3 \implies c_2 = -1 \)
\( \therefore c_1 = 4 \)

\( \therefore y(t) = (4e^{-t} - e^{-4t})u(t) \)
Zero-State Response (response due to input):
\[ \frac{d^2y}{dt^2} + 5 \frac{dy}{dt} + 4y = 1, \quad t \geq 0 \]

Since the input is constant, the particular solution (forced response) of the system is also constant. Therefore, \( \frac{dy}{dt} = \frac{d^2y}{dt^2} = 0 \)

\[ \therefore 4y_p = 1 \]
\[ \therefore y_p = \frac{1}{4} \]

Total Solution:
\[ y(t) = \left( \frac{1}{4} + 4e^{-t} + e^{-4t} \right) u(t) \]

Example 3: Repeat example 2 if \( \frac{B}{M} = 45 \) and \( \frac{K}{M} = 45^2 \).

C.E.:
\[ \lambda^2 + 4\lambda + 4 = 0 \implies \lambda_1 = -2 \text{ and } \lambda_2 = -2 \]

Zero-input response is equal to
\[ y_i(t) = (c_1 + c_2 t) e^{-2t} \]
\[ y(0) = c_1 = 3 \]
\[ y'(t) = -2c_1 e^{-2t} + c_2 e^{-2t} - 2c_2 t e^{-2t} \]
\[ \therefore y(0) = -2c_1 + c_2 = 0 \implies c_2 = 6 \]
\[ \therefore y_i(t) = (3 + 6t) e^{-2t} u(t) \]
and the particular solution \( y_p = \frac{1}{4} \)

\[ \therefore y(t) = \left( \frac{1}{4} + (3 + 6t) e^{-2t} \right) u(t) \]
Example 4: Repeat example 2 \( y \frac{B}{M} = 25^2 \) and \( \frac{K}{M} = 10^2 \).

C.E. \( \lambda^2 + 2\lambda + 10 = 0 \)

\[ \lambda_{1,2} = \frac{-2 \pm \sqrt{4 - 40}}{2} = -1 \pm j3 = \sigma \pm j\omega \]

The homogeneous solution (response due to initial conditions) is

\[ y(t) = (C_1 \cos \omega t + C_2 \sin \omega t) e^{-\sigma t} \quad ; \quad t \geq 0 \]

\[ = (C_1 \cos 3t + C_2 \sin 3t) e^{-t} \quad ; \quad t \geq 0 \]

\[ y(0) = 3 = C_1 \]

\[ y(0) = -C_1 + 3C_2 = 0 \quad \Rightarrow \quad C_2 = 1 \]

\[ y(t) = (3 \cos 3t + \sin 3t) e^{-t} \quad , \quad t \geq 0 \]

\[ = M e^{-t} \cos (3t + \theta) \]

\[ 3 \cos 3t + \sin 3t = M \cos (3t + \theta) \]

\[ = M (\cos 3t \cos \theta - \sin 3t \sin \theta) \]

\[ \Rightarrow \quad 3 = M \cos \theta \quad -1 = M \sin \theta \quad \Rightarrow \quad M = \sqrt{3^2 + 1^2} = \sqrt{10} \]

\[ \theta = -\arctan \left( \frac{1}{3} \right) \]

\[ = -18.4^\circ \]

\[ y(t) = \sqrt{10} e^{-t} \cos (3t - 18.4^\circ) u(t) \]

\[ \Rightarrow \quad Total \ response = (\frac{3}{\sqrt{10}} e^{-t} \cos (3t - 18.4^\circ)) u(t) \]