Solving Differential Equations in Engineering

Differential equations relate an output variable $y(t)$ and its derivatives to some input function $f(t)$, i.e.,

$$A_n \frac{d^n y}{dt^n} + A_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \ldots + A_1 \frac{dy}{dt} + A_0 y(t) = f(t),$$  

(1)

where the coefficients $A_n, A_{n-1}, \ldots, A_0$ can be constants, functions of $y$ or functions of $t$. The input function $f(t)$ (also called the “forcing function”) represents everything on the right hand side of the differential equation. The solution to the differential equation is the output variable, $y(t)$.

For a second order system involving position $y(t)$, velocity $\frac{dy}{dt}$ and acceleration $\frac{d^2 y}{dt^2}$, equation (1) takes the form

$$A_2 \frac{d^2 y}{dt^2} + A_1 \frac{dy}{dt} + A_0 y(t) = f(t).$$  

(2)

Note that engineers often use a “dot” notation when referring to derivatives with respect to time, i.e., $\dot{y} = \frac{dy}{dt}$, $\ddot{y} = \frac{d^2 y}{dt^2}$, etc. In this case, equation (2) can be written as

$$A_2 \ddot{y} + A_1 \dot{y} + A_0 y(t) = f(t).$$  

(3)

In many engineering applications, the coefficients $A_n, A_{n-1}, \ldots, A_0$ are constants (not functions of $y$ or $t$). For example, in the case of a spring-mass system subjected to an applied force $f(t)$, the governing differential equation is

$$m\ddot{y} + ky = f(t),$$  

(4)

where $m$ is the mass and $k$ is the spring constant.

If the coefficients $A_n, A_{n-1}, \ldots, A_0$ are functions of $y$ or $t$, exact solutions can be difficult to obtain. In many cases exact solutions do not exist, and the solution $y(t)$ must be obtained numerically (e.g., using the differential equation solvers in MATLAB). However, in the case of constant coefficients, the solution $y(t)$ to a differential equation of any order can be obtained by following the step-by-step procedure outlined below.

1. **Find the Transient Solution, $y_{trans}(t)$ (also called the “Homogeneous” or “Complementary” Solution):**
   a) Set the forcing function $f(t)=0$. This makes the right hand side of the equation zero:

   $$A_n \frac{d^n y}{dt^n} + A_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \ldots + A_1 \frac{dy}{dt} + A_0 y(t) = 0$$

   b) Assume a solution of the form $y(t) = e^{st}$, and substitute it into the above equation. Note that $\frac{dy}{dt} = se^{st}$, $\frac{d^2 y}{dt^2} = s^2 e^{st}$, etc., so that each term will contain an $e^{st}$. Since the right hand side of the equation is zero, canceling the $e^{st}$ will result in a polynomial in $s$,
\[ A_n s^n + A_{n-1} s^{n-1} + \ldots + A_1 s + A_0 = 0. \]

c) Solve for the roots of the above polynomial. These are the \( n \) values of \( s \) that make the polynomial equal to zero. Call these values \( s_1, s_2, \ldots, s_n \).

d) For the case of \( n \) distinct roots, the transient solution of the differential equation has the general form

\[ y_{\text{trans}}(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t} + \ldots + c_n e^{s_n t}, \]

where the constants \( c_1, c_2, \ldots, c_n \) are determined later from the initial conditions on the problem.

e) For the special case of repeated roots (i.e., two of the roots are the same), the solution can be made general by multiplying one of the roots by \( t \). For example, for a second order system with \( s_1 = s_2 = s \), the transient solution is

\[ y_{\text{trans}}(t) = c_1 e^{st} + c_2 t e^{st}. \]

2. **Find the Steady State Solution** \( y_{ss}(t) \) (also called the Particular Solution):

The steady-state solution can be found using the *Method of Undetermined Coefficients*:

a) Assume (guess) the form of the steady-state solution \( y_{ss}(t) \). This will usually have the same general form as the forcing function and its derivatives, but will contain unknown constants (i.e., undetermined coefficients). Example guesses are shown in the table below, where \( K, A, B \) and \( C \) are constants:

<table>
<thead>
<tr>
<th>If input ( f(t) ) is</th>
<th>Assume ( y_{ss}(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( K )</td>
<td>( A )</td>
</tr>
<tr>
<td>( Kt )</td>
<td>( A t + B )</td>
</tr>
<tr>
<td>( Kt^2 )</td>
<td>( A t^2 + B t + C )</td>
</tr>
<tr>
<td>( K \sin \omega ) or ( K \cos \omega )</td>
<td>( A \sin \omega + B \cos \omega )</td>
</tr>
</tbody>
</table>

b) Substitute the assumed steady state solution \( y_{ss}(t) \) and its derivatives into the *original* differential equation.

c) Solve for the unknown (undetermined) coefficients \( A, B, C \), etc.). This can usually be done by equating the coefficients of like terms on the right and left hand sides of the equation.

3. **Find the Total Solution**, \( y(t) \):

a) The total solution is just the sum of the transient and the steady state solutions,

\[ y(t) = y_{\text{trans}}(t) + y_{ss}(t). \]

b) Apply the *initial conditions* on \( y(t) \) and its derivatives. A differential equation of order \( n \) must have exactly \( n \) initial conditions, which will result in an \( n \times n \) system of equations for the \( n \) constants \( c_1, c_2, \ldots, c_n \).
Differential Equations in Engineering

Leaking Bucket Problem:

\[ Q_{in} - Q_{out} = \frac{dv}{dt} \]

\[ \Rightarrow \frac{dv}{dt} + Q_{out} = Q_{in} \]

We know \( Q_{out} = K \cdot r(t) \)

\[ \Rightarrow \frac{dv}{dt} + K \cdot r(t) = Q_{in} \]

Now, \( \frac{dv}{dt} = A \cdot \frac{dr(t)}{dt} \)

\[ \Rightarrow A \cdot \frac{dr(t)}{dt} + K \cdot r(t) = Q_{in} \]

or

\[ \frac{dr(t)}{dt} + \left( \frac{K}{A} \right) r(t) = \frac{1}{A} Q_{in} \] (1)

Equation (1) is a first-order differential equation in \( r(t) \). The solution of the differential equation consists of two components:

(i) Zero-input response, i.e., the response due to initial condition when the input = 0.

(ii) Zero state response, i.e., the response due to input when the initial condition = 0.
The total response is the sum of the two responses, i.e.,

\[ \text{Total response} = \text{zero-input response} + \text{zero-state response} \]

\[ \text{Response due to initial condition when the input } = 0 \]
\[ \text{(Also called the transient response)} \]
\[ \text{Response due to input when the initial condition } = 0 \]
\[ \text{(Also called the steady-state response)} \]

(1) Transient (complementary) solution:

For the leaking bucket problem, if \( Q_{in} = 0 \) as in the laboratory No 7, we get

\[ Q_{out} = \sum \]

\[ \frac{dP(t)}{dt} + \left( \frac{K}{A} \right) P(t) = 0 \quad \text{(Homogeneous equation)} \]

Assume \( P(t) = Ce^{st} \)

\[ \frac{dP}{dt} = Cs e^{st} \]

\[ Cs e^{st} + \left( \frac{K}{A} \right) Ce^{st} = 0 \]

\[ (s + \frac{K}{A})Ce^{st} = 0 \]

\[ s + \frac{K}{A} = 0 \quad \text{(Characteristic equation)} \]

\[ s = -\frac{K}{A} \]

\[ P(t) = C e^{-\left( \frac{K}{A} \right)t} \]
At \( t = 0 \)
\[
P(0) = C e^0 = C
\]
\[
\therefore \quad P(t) = P(0) e^{\left(\frac{A}{K}\right)t}
\] (2)

Equation (2) is the response of the differential equation if the input \( Q_{in} = 0 \) and the initial condition (initial height) \( P(0) \). Equation (2) gives the transient response of the system.

\[
\therefore \quad P_{\text{transient}}(t) = P(0) e^{\left(\frac{A}{K}\right)t} = P(0) e^{\frac{A}{K}t}
\]

\[\text{time constant} = \frac{A}{K}\]

\( P(0) \)

Diagram:

\( 0.368 \times P(0) \)

\( t = 0 \)

\( \frac{A}{K} = 2 \)

(2) **Steady-State (Particular) Solution:**

The steady-state solution is the solution to a particular input. For the leaking bucket problem, let \( Q_{in} = B = \text{Constant} \).

\[
\therefore \quad \frac{dP(t)}{dt} + \frac{K}{A} P(t) = \frac{1}{A} B \quad \text{(non-homogeneous)}
\]

* The particular solution of a given non-homogeneous differential equation (when R.H.S \( \neq 0 \), let R.H.S = 800) can be found using the method of undetermined coefficients.*
The method of undetermined coefficients relies on inspecting the expression \( g(x) \) and determining a combination of all possible functions that, after differentiating, would result in at least one of the term in the expression \( g(x) \).

The following table gives examples of \( g(x) \) and its corresponding trial solution of particular or steady-state solution \( y_{ss} \):

<table>
<thead>
<tr>
<th>( y(x) )</th>
<th>( y_{ss}(x) )</th>
<th>Try</th>
<th>Yes</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( \text{Constant, } a )</td>
<td>( \text{Constant, } A )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2. ( \alpha x^n )</td>
<td>( A_0 x^0 + A_1 x^1 + \ldots + A_n x^n + A_0 e^{ax} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3. ( e^{ax} )</td>
<td>( A e^{ax} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4. ( \alpha \cos bx )</td>
<td>( \alpha \cos bx + B \sin bx )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5. ( \alpha \sin bx )</td>
<td>( \alpha \cos bx + B \sin bx )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Now, for the leaky bucket problem, guess \( y_{ss}(t) = E \)

\[
\frac{\text{d}y_{ss}(t)}{\text{d}t} = 0
\]

or \( 0 + \frac{K}{A} E = \frac{1}{A} B \)

\[
E = \frac{B}{K}
\]

\( y_{ss}(t) = \frac{B}{K} \)

(3) Total Solution:

\[ y(t) = y_{\text{trans}}(t) + y_{ss}(t) \]
\[ P(t) = P(0) e^{-\frac{(K/2)}{K}} + \frac{B}{K} \]

Note: as \( t \to \infty \), \( P(t) = \frac{B}{K} = \text{steady state solution} \)

At steady state, \( P = \text{constant} \)

* Now, suppose \( P(0) = 0 \)

\[ P(0) = C e^0 + \frac{B}{K} = 0 \]

\[ C = -\frac{B}{K} \]

\[ \text{Total Solution} \]

\[ P(t) = -\frac{B}{K} e^{\frac{K}{2} t} + \frac{B}{K} \]

\[ = \frac{B}{K} (1 - e^{\frac{K}{2} t}) \]

At \( t = \frac{A}{K} \)

\[ K = \frac{B}{K} (1 - e^{\frac{K}{2} t}) \approx 0.632 \frac{B}{K} \]

* Physically, the container continues to fill until the pressure is great enough that \( \frac{dV}{dt} = 0 \) i.e. \( \frac{dV}{dt} = 0 \) ! That value depends on \( \frac{B}{K} = \frac{q_{out}}{K} \), where \( K \) is a function of fluid properties and outlet geometry.
Differential Equations in Mechanical Systems:
(ME 213, 414, 460)

Example: Free Vibration of a Spring-Mass System

1) Free-body diagram at equilibrium ($y = 0$)

\[ \Sigma F_y = 0 \Rightarrow ks - Mg = 0 \]

or $S = Mg/k$

2) Free-body diagram at $y > 0$

\[ \Sigma F_y = Ma = My' \]

\[ Mg - k(s + y(t)) = My' \]
\[ M\ddot{y} - Ky(t) = M\ddot{y} \]
\[ M\ddot{y} - K\left(\frac{Ma}{K}\right) - Ky(t) = M\ddot{y} \]

\[ M\ddot{y} + Ky = 0 \quad \text{or} \quad \ddot{y} + \frac{K}{M}y = 0 \]

**General solution**

1) Transient (Complementary) Solution

Assume \( y = Ce^{st} \)

\[ \ddot{y} = cs^2e^{st} \]
\[ \dot{y} = cse^{st} \]
and \( y = ce^{st} \)

Substituting \( \dot{y} \) in the differential equation

\[ cs^2e^{st} + \left(\frac{K}{M}\right)ce^{st} = 0 \]
\[ ce^{st}\left(s^2 + \frac{K}{M}\right) = 0 \]

\( s^2 + \frac{K}{M} = 0 \) (Characteristic equation)

\[ s = \pm j\sqrt{\frac{K}{M}} \]

Two roots are \( s_1 = \frac{j}{\sqrt{\frac{K}{M}}} \) and \( s_2 = -\frac{j}{\sqrt{\frac{K}{M}}} \)

\[ y_{\text{tran}}(t) = C_1e^{s_1t} + C_2e^{s_2t} \]
\[ = C_1e^{\frac{j}{\sqrt{\frac{K}{M}}}t} + C_2e^{-\frac{j}{\sqrt{\frac{K}{M}}}t} \]

Recall from complex numbers,

\[ e^{j\theta} = \cos \theta + j \sin \theta \] (Euler's formula)

\[ e^{-j\theta} = \cos \theta - j \sin \theta \]
\[ y_{\text{trans}}(t) = C_1 (\cos \sqrt{\frac{K}{M}} t + j \sin \sqrt{\frac{K}{M}} t) + C_2 (\cos \sqrt{\frac{K}{M}} t - j \sin \sqrt{\frac{K}{M}} t) \]
\[ = (C_1 + C_2) \cos \sqrt{\frac{K}{M}} t + j (C_1 - C_2) \sin \sqrt{\frac{K}{M}} t \]
\[ = C_3 \cos \sqrt{\frac{K}{M}} t + C_4 \sin \sqrt{\frac{K}{M}} t \]

**Note:** If the roots of the characteristic equation are complex, we can directly write the solution in terms of sine and cosine of the imaginary part of the roots, i.e., if the roots are \( s = \pm \omega t \), the transient response can be written as
\[ y_{\text{trans}}(t) = p_1 \sin \omega t + p_2 \cos \omega t \]

*Evaluation of the constant \( C_3 \) and \( C_4 \) using the initial conditions.*

Let \( y(0) = A \) and \( \dot{y}(0) = 0 \)

Since \( y_{\text{trans}}(t) = C_3 \cos \sqrt{\frac{K}{M}} t + C_4 \sin \sqrt{\frac{K}{M}} t \)
\[ y(0) = A = C_3 \cos (0) + C_4 \sin (0) = C_3 \]
\[ \therefore C_3 = A \]
\[ \dot{y}(t) = C_3 (-\sqrt{\frac{K}{M}} \sin \sqrt{\frac{K}{M}} t) + C_4 (\sqrt{\frac{K}{M}} \cos \sqrt{\frac{K}{M}} t) \]
\[ \dot{y}(0) = 0 = -\sqrt{\frac{K}{M}} C_3 \sin (0) + \sqrt{\frac{K}{M}} C_4 \cos (0) \]
\[ C_4 \sqrt{\frac{K}{M}} = 0 \]
\[ \Rightarrow C_4 = 0 \]

\[ y(t) = A \cos \left( \sqrt{\frac{K}{M}} t \right) \]

2) Steady-state solution:

Since R.H.S of differential equation = 0

\[ y_{ss}(t) = 0 \quad \text{(no forcing function)} \]

3) Total solution:

\[ y(t) = y_{trans}(t) + y_{ss}(t) \]

\[ = A \cos \sqrt{\frac{K}{M}} t \]

\[ = A \cos \omega_n t \]

Where \( \omega_n = \sqrt{\frac{K}{M}} \) rad/s is the natural frequency. The natural frequency, \( \omega_n \), increases with stiffness \( K \) of the spring and decreases with mass.

![Graph](image)
Example: The mass-spring system is subjected to an applied force $F \cos \omega t$ as shown.

\[ \begin{align*}
\Sigma F_y &= M \ddot{y} \\
F \cos \omega t - Ky &= M \ddot{y} \\
\therefore M \ddot{y} + Ky &= F \cos \omega t
\end{align*} \]

1) Transient Solution:
   Characteristic equation:
   \[ S^2 + \frac{K}{M} = 0 \]
   \[ S = \pm \sqrt{-\frac{K}{M}} \]
   \[ y_{\text{trans}}(t) = C_1 e^{\sqrt{-\frac{K}{M}} t} + C_2 e^{\sqrt{-\frac{K}{M}} t} \]

2) Steady-State Solution:
   Since the forcing function is $f(t) = F \cos \omega t$
   Assume $y_{\text{ss}}(t) = A \sin \omega t + B \cos \omega t$
\[ \dot{y}_{ss}(t) = A w_0 \cos w_0 t - B w_0 \sin w_0 t \]

and \[ \ddot{y}_{ss}(t) = -A w_0^2 \sin w_0 t - B w_0^2 \cos w_0 t \]

Substituting \( y_{ss}(t) \) and \( \ddot{y}_{ss}(t) \) into the differential equation, we get

\[-w_0^2 (A \sin w_0 t + B \cos w_0 t) + \frac{F}{M} (A \sin w_0 t + B \cos w_0 t) = \frac{F}{M} \cos w_0 t \]

or

\[(\frac{K}{M} - w_0^2) A \sin w_0 t + (\frac{K}{M} - w_0^2) B \cos w_0 t = \frac{F}{M} \cos w_0 t \]

Comparing the coefficients of \( \sin w_0 t \) on both sides, we get

\[(\frac{K}{M} - w_0^2) A = 0 \quad \Rightarrow \quad A = 0 \]

Comparing the coefficients of \( \cos w_0 t \) on both sides, we get

\[(\frac{K}{M} - w_0^2) B = \frac{F}{M} \]

\[ \Rightarrow \quad B = \frac{F}{K - M w_0^2} \]

\[ \therefore \quad \ddot{y}_{ss}(t) = \left( \frac{F}{K - M w_0^2} \right) \cos w_0 t \]

3) Total Solution: \( y(t) = y_{ss}(t) + \dot{y}_{ss}(t) \)

\[ \therefore \quad y(t) = C_3 \cos \sqrt{\frac{K}{M}} t + C_4 \sin \sqrt{\frac{K}{M}} t + \left( \frac{F}{K - M w_0^2} \right) \cos w_0 t \]

* Solving the constants \( C_3 \) and \( C_4 \) using the initial conditions: \( y(0) = 0 \) and \( \dot{y}(0) = 0 \)

\[ y(0) = 0 = C_3 \cos 0 + 0 + \left( \frac{F}{K - M w_0^2} \right) \cos 0 \]

\[ \therefore \quad C_3 = - \frac{F}{K - M w_0^2} \]
\[ \dot{y}(t) = -\sqrt{\frac{K}{M}} C_3 \sin \sqrt{\frac{K}{M}} \, t + \sqrt{\frac{K}{M}} C_4 \cos \sqrt{\frac{K}{M}} \, t + \left(\frac{-F_0}{K - M \omega_0^2}\right) \sin \omega_0 t \]

\[ \dot{y}(t) = 0 = \quad 0 + \sqrt{\frac{K}{M}} C_4 \cos 0 + 0 \]

\[ = \sqrt{\frac{K}{M}} C_4 \]

or \( C_4 = 0 \)

\[ \therefore y(t) = \left(\frac{F_0}{K - M \omega_0^2}\right) \left( \cos \omega_0 t - \cos \sqrt{\frac{K}{M}} \, t \right) \]

b) The value of \( y(t) \) as \( \omega_0 \rightarrow \sqrt{\frac{K}{M}} \)

\[ y(t) = \left(\frac{F_0}{K - M \omega_0^2}\right) \left( \cos \sqrt{\frac{K}{M}} \, t - \cos \sqrt{\frac{K}{M}} \, t \right) = \frac{F_0}{K - M \omega_0^2} \]

This is an "indeterminate" form, and can be evaluated by methods of calculus not yet available to all students.

Instead, the result can be investigated by picking a value of \( \omega_0 \) close to \( \sqrt{\frac{K}{M}} \) and plotting the result.

(e.g., let \( K = M = F = 1 \) and choose the values of \( \omega_0 \)

\( \omega_0 = 0.9 \sqrt{\frac{K}{M}}, 0.99 \sqrt{\frac{K}{M}} \) and \( 0.999 \sqrt{\frac{K}{M}} \)
The first plot shows the "beating" phenomenon typical of problems where the forcing frequency \( \omega \) is in the neighborhood of the natural frequency \( \sqrt{k/m} \). As \( \omega \) is increased to 0.99 \( \sqrt{k/m} \) and 0.9999 \( \sqrt{k/m} \), the last two plots show \( y(t) \) increasing without bound. This is called resonance, and is generally undesirable.