Graphing the Sine and Cosine Functions

Example: For the one-link robot, plot the x- and y-components of the tip of the robot as θ goes through one revolution, i.e., varies from 0 to 2π radians.

Solution:

Note: The y-component is the plot of \( l \sin \theta \) as \( \theta \) varies from 0 to 2π radians.
- \( l \) is the amplitude or magnitude.
- \( \sin \theta \) starts at 0 and goes up to 1. It then wiggles down to -1 and back to 0 as \( \theta \) completes one revolution.
- The x-component is the plot of $l \cos \theta$ as $\theta$ varies from 0 to $2\pi$ radians.
- $l$ is the amplitude.
- $\cos \theta$ starts at 1 and then goes to 0 to -1 and back to 0 and finish at 1 as $\theta$ completes one revolution, i.e., goes from 0 to $2\pi$ radians.

* Both sine and cosine will repeat the cycle as $\theta$ varies from $0 \pi$ to $4\pi$, $4\pi$ to $6\pi$ and so on.

**Therefore, both sine and cosine are periodic with a period of $2\pi$**

\[
\sin(\theta + 2\pi) = \sin \theta \\
\cos(\theta + 2\pi) = \cos \theta
\]

**Note:**
- $2\pi$ radian = 360°
- $\pi$ radian = 180°
- 1 radian = 57.3°

**Example:** For the one-link robot, the tip is at $\theta = 0$ at time $t = 0$ sec. It takes the tip $2\pi$ sec to go from 0 to $2\pi$ radians. Plot the x- and y-components of the tip as a function of time. Also, find the amplitude, frequency, and period of the sine and cosine functions.
Let the unit sine and cosine as
\[ x = \ell \cos \omega t = \ell \cos \theta \]
and \[ y = \ell \sin \omega t = \ell \sin \theta \]
\[ \therefore \theta = \omega t \]
where \( \omega \) is in rad/s.

Thus, \[ \omega = \frac{\theta}{t} = \frac{2\pi}{2\pi} = 1 \text{ rad/s} \]
Therefore, the frequency is 1 rad/s. This frequency is also known as the angular frequency.
The robot completes 1 cycle (θ = 0 to 2π) in 2π sec or 1/2π cycles in 1 sec. This frequency is known as the linear frequency and is designated as \( f \).

\[
\therefore f = \frac{1}{2\pi} \text{ c/s} = \frac{w}{2\pi} \text{ c/s or Hertz (Hz)}
\]

Thus, the relationship between the angular frequency \( (w) \) and the linear frequency \( (f) \) is given by

\[
w = 2\pi f
\]

The amplitudes of the sine and cosine functions are given by \( b \).

\[\text{Period, } T = \frac{2\pi}{w} \text{ sec} = \frac{2\pi}{w} \text{ sec} \]

\[
\therefore T = \frac{2\pi}{w} \quad \text{or} \quad w = \frac{2\pi}{T}
\]

Where \( T \) is the period in sec and \( w \) is the angular frequency (or frequency) in rad/s.
Example: For the one-link robot, the tip is at \( \theta = \frac{\pi}{8} \) rad at time \( t = 0 \) sec. It takes the tip 1 sec to go from \( \frac{\pi}{8} \) to \( \frac{3\pi}{8} \) rad. Plot the \( x \)- and \( y \)-components of the tip as a function of time. Also, find the amplitude, frequency and period.

Amplitude = 10
Frequency \( \omega = \frac{2\pi}{T} = \frac{2\pi}{1} \text{ rad/s} \)
Period = 1 Sec

\( x \)- and \( y \)-components can be written as:
\[
x = 10 \cos \left( 2\pi t + \frac{\pi}{8} \right)
\]
\[
y = 10 \sin \left( 2\pi t + \frac{\pi}{8} \right)
\]
A general sinusoidal function, therefore, can be written as

\[ V(t) = V_m \sin(\omega t + \theta) \]

Where \( V_m \) is the amplitude (magnitude), \( \omega \) is the frequency in rad/s and \( \theta \) is the phase angle or simply phase.

* To be consistent, since \( \omega t \) is in radians, \( \theta \) should be in radians. However, degrees are a very familiar measure for angles. Therefore, we may write

\[ x = 10 \cos\left(2\pi t + \frac{\pi}{8}\right) \]

\[ = 10 \cos\left(2\pi t + 22.5\right) \]

interchangeably, even though the latter expression contains a formal mathematical inconsistency.

* In the absence of a small circle indicating degrees, the unit for angle will be taken as radians.

Note: 

<table>
<thead>
<tr>
<th>T = \frac{2\pi}{\omega}</th>
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<tbody>
<tr>
<td>\omega = 2\pi f</td>
</tr>
<tr>
<td>f = \frac{1}{T}</td>
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Example: Figure below shows a mass \( M \) attached to a spring which is fixed at the other end. When the mass is pulled and released, the mass does not simply return to its equilibrium point at the starting position. The object actually oscillates above and below the equilibrium point and the displacement is given by

\[
y = 2 \sin (6\pi t + \frac{\pi}{2})
\]

This oscillation is called simple harmonic motion.

(a) Find the amplitude, frequency, period and the phase angle.
(b) Determine the displacement of the system 2 sec after release.
(c) Determine when the object will first reach its maximum negative displacement.
(d) Plot the displacement as a function of time.
Solution:

(a) \[ y = 2 \sin(6\pi t + \frac{\pi}{2}) \]

Amplitude = 2
Angular frequency, \( \omega = 6\pi \text{ rad/s} \)
Linear frequency, \( f = \frac{\omega}{2\pi} = 3 \text{ C/s or Hz} \)
Period, \( T = \frac{2\pi}{\omega} = \frac{1}{f} = \frac{1}{3} \text{ sec} \)
Phase angle = \( \frac{\pi}{2} \text{ rad} = 90^\circ \)
Phase Shift = \( \frac{\pi/2}{2\pi} = \frac{\theta}{\omega} = \frac{1}{12} \text{ sec} \)

(b) \[ y(3 \text{ sec}) = 2 \sin \left(6\pi(3) + \frac{\pi}{2}\right) \]
\[ = 2 \sin \left(18\pi + \frac{\pi}{2}\right) \]
\[ = 2 \sin \left(\frac{37\pi}{2}\right) \]
\[ = 2 \sin \left(18 \cdot 5\pi - 18\pi\right) \]
\[ = 2 \sin \left(0 \cdot 5\pi\right) \]
\[ = 2 \sin \left(\frac{\pi}{2}\right) = 2 \]

Maximum height.

(c) The maximum negative displacement for the regular sine function \( y = a \sin t \)
occurs at \( t = \frac{3\pi}{2} \), therefore

\[ 6\pi t + \frac{\pi}{2} = \frac{3\pi}{2} \]

\[ \therefore 6\pi t = \pi \]
\[ \therefore t = \frac{\pi}{6\pi} = \frac{1}{6} \text{ sec} \]
Example: If the spring mass system shown in the previous example is pulled 5 cm from its equilibrium position and released at t=0. Oscillating with simple harmonic motion, it takes the system 2 sec to return to the point of release. Write the equation describing the displacement.

Solution: \[ y = d \sin(\omega t + \theta) \text{ cm} \]
\[ d = 5 \]
\[ \omega = \frac{2\pi}{2} = \pi \text{ rad/s} \]
At \( t = 0 \), \( y = d = 5 \text{ cm} \)
\[ \therefore 5 = 5 \sin(0 + \theta) \]
\[ \therefore \sin \theta = 1 \Rightarrow \theta = \frac{\pi}{2} \]
\[ \therefore y = 5 \sin(\pi t + \frac{\pi}{2}) \text{ cm} \]
Example: An alternating current has a peak current of 0.5 A and a frequency of 50 Hz (Hertz, c/s). If the phase angle is 45°, write the equation of current as a function of time. Also, find the period and the current at t = 0.2 sec.

Solution:

\[ i(t) = I_m \sin(\omega t + \theta) \text{ A} \]

\[ I_m = 0.5 \]

\[ \omega = 2\pi f = 2\pi (50) = 100\pi \text{ rad/s} \]

\[ \theta = 45^\circ = \frac{\pi}{4} \text{ rad} \]

\[ \therefore i(t) = 0.5 \sin \left(100\pi t + \frac{\pi}{4}\right) \text{ A} \]

\[ T = \frac{2\pi}{\omega} = \frac{2\pi}{100\pi} = \frac{1}{50} \text{ sec} \]

\[ i(0.2) = 0.5 \sin \left(100\pi (0.2) + \frac{\pi}{4}\right) \]

\[ = 0.5 \sin \left(20\pi + \frac{\pi}{4}\right) \]

\[ = 0.5 \sin \left(\frac{\pi}{4}\right) \]

\[ = 0.5 \times 0.707 \]

\[ = 0.3535 \text{ A} \]
Example: A current of $6 \sin at$ is applied to a circuit containing a resistor of $1 \Omega$ and an inductor of $\frac{3}{8} H$. The voltage across the resistor is given by $6 \sin at$ and the voltage across the inductor is given by $8 \cos at$. Find the total voltage across the circuit if we know that $v(t) = v_R + v_L = 6 \sin at + 8 \cos at$.

Solution: Using KVL

$$v(t) = v_R + v_L$$

$$= 6 \sin at + 8 \cos at$$

Note: The sum of a sine function and a cosine function of the same frequency is another sinusoid (sine or cosine) of the same frequency, i.e.

$$v(t) = 6 \sin at + 8 \cos at$$

$$= M \sin(at + \theta)$$

Therefore, all we need to do is to find $M$ and $\theta$. 
We know
\[ M \sin(\omega t + \theta) = M(\sin \omega t \cos \theta + \cos \omega t \sin \theta) = (M \cos \theta) \sin \omega t + (M \sin \theta) \cos \omega t \]

Therefore, we can write
\[ 6 \sin \omega t + 8 \cos \omega t = (M \cos \theta) \sin \omega t + (M \sin \theta) \cos \omega t \]

Comparing the coefficients of \( \cos \omega t \), we get
\[ M \cos \theta = 8 \]

Comparing the coefficients of \( \sin \omega t \), we get
\[ M \sin \theta = 6 \]

Therefore, we have
\[ \frac{M}{\sin \theta} = \frac{8}{6} = \frac{4}{3} \]

\[ \sin \theta = \frac{3}{4} \]

\[ \cos \theta = \frac{4}{4} = \frac{1}{4} \]

\[ M = \sqrt{6^2 + 8^2} = 10 \]

\[ \theta = \tan^{-1} \left( \frac{6}{8} \right) = \tan^{-1} \left( \frac{3}{4} \right) = 53.3^\circ \]

\[ V(t) = 10 \sin(\omega t + 53.3^\circ) \text{ volts} \]

\[ \therefore \text{Voltage leads the current by } 53.3^\circ \]
Note:
\[ A \cos \omega t + B \sin \omega t = \sqrt{A^2 + B^2} \sin(\omega t + \arctan \frac{B}{A}) \]
and
\[ A \cos \omega t + B \sin \omega t = \sqrt{A^2 + B^2} \cos(\omega t - \arctan \frac{A}{B}) \]

Example: Find \( M \) and \( \theta \) for the expression

\[ -5 \cos 3t + 12 \sin 3t = M \cos (3t + \theta) \]

Solve:

\[ -5 \cos 3t + 12 \sin 3t = M \cos 3t \cos \theta - \sin 3t \sin \theta \]

\[ = (M \cos \theta) \cos 3t + (-M \sin \theta) \sin 3t \]

Compare the coefficients of \( \sin 3t \) on both sides,

\[ -M \sin \theta = 12 \]

Compare the coefficient of \( \cos 3t \) on both side

\[ M \cos \theta = -5 \]

\[ \therefore M \sin \theta = -12 \]

\[ M \cos \theta = -5 \]

\[ \therefore -5 \sin 3t + 12 \sin 3t = 13 \cos (3t - 112.6^\circ) \]
Systems of Equation

Example: The following system of equations results from the Kirchhoff's laws to the circuit shown. Solve for the currents $I_1$ and $I_2$.

\[ \begin{align*}
6I_1 + 4I_2 &= 6 \\
4I_1 + 12I_2 &= 9 
\end{align*} \]

Solution: The two equations with two unknowns $I_1$ and $I_2$ can be solved using four different methods.

1. Substitution Method
2. Graphical Method
3. Matrix Algebra
4. Cramer's Rule

1. Substitution Method

Solve $I_1$ from the first equation

\[ \therefore 10I_1 = 6 - 4I_2 \]

\[ \therefore I_1 = \frac{6 - 4I_2}{10} \]
Substitute \( I_1 = \frac{6-4I_2}{10} \) into the second equation

\[
4 \left( \frac{6-4I_2}{10} \right) + 12I_2 = 9
\]

\[
2.4 - 1.6I_2 + 12I_2 = 9
\]

\[
\Rightarrow 10.4I_2 = 6.6
\]

\[
I_2 = \frac{6.6}{10.4} = 0.6346 \text{ A}
\]

and \( I_1 = \frac{6-4(0.6346)}{10} \approx 0.3462 \text{ A} \)

2. **Graphical Method**:

Draw two straight lines (one for each equation) with \( I_1 \) as the x-axis and \( I_2 \) as the y-axis.

From the first equation: \( I_2 = -2.5I_1 + 1.5 \)

From the second equation: \( I_2 = -\frac{3}{4}I_1 + \frac{3}{4} \)

From the intersection: \((0.346, 0.634) = (I_1, I_2)\)
3. Matrix Algebra:

The two equations with two unknown currents can be written in the matrix form as

\[
\begin{bmatrix}
10 & 4 \\
4 & 12 \\
\end{bmatrix}
\begin{bmatrix}
I_1 \\
I_2 \\
\end{bmatrix}
= 
\begin{bmatrix}
6 \\
9 \\
\end{bmatrix}
\]

\[A \begin{bmatrix}
I \\
\end{bmatrix} = \begin{bmatrix}
B \\
\end{bmatrix}\]

Where \(A\) is a 2x2 (2 rows and 2 columns) matrix, \(I\) is a 2x1 (column) unknown vector, and \(B\) is a 2x1 known vector.

\[
A = \begin{bmatrix}
10 & 4 \\
4 & 12 \\
\end{bmatrix}, \quad \begin{bmatrix}
I_1 \\
I_2 \\
\end{bmatrix}, \quad \begin{bmatrix}
6 \\
9 \\
\end{bmatrix}
\]

\[
A = \begin{bmatrix}
10 & 4 \\
4 & 12 \\
\end{bmatrix}
\]

\[
I = \begin{bmatrix}
I_1 \\
I_2 \\
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
6 \\
9 \\
\end{bmatrix}
\]

The unknown \(I\) can be solved as

\[
I = A^{-1}B
\]

where \(A^{-1}\) is the inverse of the matrix \(A\).

To find the inverse of a 2x2 matrix

\[
A = \begin{bmatrix}
a & b \\
c & d \\
\end{bmatrix}
\]

use the following formula:

\[
A^{-1} = \frac{1}{\Delta} \begin{bmatrix}
d & -b \\
-c & a \\
\end{bmatrix}
\]

where \(\Delta = ad - bc\).
where $\Delta$ is the determinant of matrix $A$. If $\Delta = 0$, then $A$ does not have a \textit{true} inverse.

$$\Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

\text{mins} \quad \text{Add}

Now, let us find the inverse of matrix $A$

$$A = \begin{bmatrix} 10 & 4 \\ 4 & 12 \end{bmatrix}$$

$$\Delta = \begin{vmatrix} 10 & 4 \\ 4 & 12 \end{vmatrix} = 120 - 16 = 104$$

$$\therefore \ A^{-1} = \frac{1}{\Delta} \begin{bmatrix} 12 & -4 \\ -4 & 10 \end{bmatrix}$$

$$= \frac{1}{104} \begin{bmatrix} 12 & -4 \\ -4 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{26} & -\frac{1}{26} \\ -\frac{1}{26} & \frac{5}{52} \end{bmatrix}$$

$$\therefore I = A^{-1}B = \begin{bmatrix} \frac{3}{26} & -\frac{1}{26} \\ -\frac{1}{26} & \frac{5}{52} \end{bmatrix} \begin{bmatrix} 6 \\ 9 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{26} \times 6 - \frac{1}{26} \times 9 \\ -\frac{1}{26} \times 6 + \frac{5}{52} \times 9 \end{bmatrix} = \begin{bmatrix} \frac{18 - 9}{26} \\ \frac{-18 + 45}{52} \end{bmatrix}$$

$$\therefore [I_1] = \begin{bmatrix} 0.3462 \\ 0.6546 \end{bmatrix}$$
4. Cramer's Rule:

\[ \mathbf{I}_1 = \frac{\begin{vmatrix} 7 \times 4 \\ 9 \times 12 \\ 10 \times 4 \\ 4 \times 12 \end{vmatrix}}{104} = \frac{72 - 36}{104} = \frac{36}{104} = 0.3462 \]

\[ \mathbf{I}_2 = \frac{\begin{vmatrix} 10 \times 6 \\ 4 \times 9 \\ 10 \times 4 \\ 4 \times 12 \end{vmatrix}}{104} = \frac{90 - 24}{104} = \frac{66}{104} = \frac{33}{52} = 0.6346 \]

Note: You must write the equations with the unknown variable properly aligned to use Cramer's Rule:

\[ a_{11}x_1 + a_{12}x_2 = y_1 \]

\[ a_{21}x_1 + a_{22}x_2 = y_2 \]

Unknown \( x_1 \) \( \text{and} \) \( x_2 \) are unknowns

\[ \text{Unknown} \ x_2 \]

Constants on the right-hand side

Where \( x_1 \) and \( x_2 \) are unknowns

Use Cramer's Rule:
Example: A 12.0-ft beam weighing 50.0 lb is supported on each end. The uniform beam supports a 75.0-lb pile of brick 4.0 ft from the right end and a 160-lb man 3.0 ft from the left end.

A system of equations that result from this situation is

\[ A + B = 285 \]
\[ 6A - 6B = 330 \]

where \( A \) represents the force supported by the left end of the beam and \( B \) represents the force supported by the right end. Find how much weight each end supports.

Solution: We will find the solution of the two simultaneous equations in two unknowns \( A \) and \( B \) using the five methods described in the previous example.
Example: Cables support a crate as shown. Find the tension in the two cables if the equations are

\[
0.707 T_1 = 0.9 T_2 \\
0.707 T_1 + 0.5 T_2 = 95
\]

Solution:

\[
0.707 T_1 - 0.9 T_2 = 0 \\
0.707 T_1 + 0.5 T_2 = 95
\]

Using Cramer's rule

\[
T_1 = \frac{\begin{vmatrix} 0 & -0.9 \\ 95 & 0.5 \end{vmatrix}}{\begin{vmatrix} 0.707 & -0.9 \\ 0.707 & 0.5 \end{vmatrix}} = \frac{0+85.5}{.3535+.1363} = 86.4 \text{ N}
\]

\[
T_2 = \frac{\begin{vmatrix} 0 & 95 \\ 0.707 & -0.9 \end{vmatrix}}{\begin{vmatrix} 0.707 & 0 \\ 0.707 & 95 \end{vmatrix}} = \frac{0.707 \times 95 - 0}{.9898} = 67.9 \text{ N}
\]