THE INFLUENCE OF BOUNDARY CONDITIONS AND ASPECT RATIO ON APPROXIMATE SOLUTIONS FOR CONSTRAINED LAYER DAMPING TREATMENTS ON BEAMS AND PLATES

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science in Engineering

By

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B.S., Wright State University, 2001

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Wright State University
I HEREBY RECOMMEND THAT THE THESIS PREPARED UNDER MY SUPERVISION BY Brian D. Runyon ENTITLED The Influence of Boundary Conditions and Aspect Ratio on Approximate Solutions for Constrained Layer Damping Treatments on Beams and Plates BE ACCEPTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF Master of Science in Engineering.

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Abstract


Recent investigations of traditional modeling methods for constrained layer damping treatments have led to a concentrated effort to determine the effectiveness of both analytical and finite element modeling. The focus of this work is to investigate the influence of boundary conditions and aspect ratio on the ability of models to characterize the loss factor and frequency of viscoelastically damped beams and plates with constrained layer damping treatments. This research will extend previous work by allowing predictions and characterization of damped beams or plates in a more timely and accurate manner. Modal Strain Energy and complex eigenvalue solvers will be used to demonstrate the value of finite element modeling as applied to vibration damping. The value of analytical models (including 6th order beam theory), finite element modeling, and the Ross, Ungar, and Kerwin theory, will also be demonstrated. Numerical results for analytical models and finite element solutions are presented for various modes, aspect ratios, boundary conditions, and for two values of constraining layer thickness. The results of this thesis are of interest to those who characterize viscoelastic material properties and/or designers of vibration damping treatments.
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$\beta_n L$  
Eigenvalue of $n^{th}$ Vibratory Mode

$\gamma$  
Shear Strain

$\varepsilon$  
Strain

$\eta$  
Loss Factor

$\nu$  
Poisson’s Ratio

$\rho$  
Density

$\sigma$  
Stress

$\tau$  
Shear Stress

$\omega$  
Circular Frequency

$A$  
Area

$B$  
Flexural Rigidity

$D$  
Displacement of Neutral Axis

$D_i$  
Bending Stiffness of $i^{th}$ Layer

$E$  
Young’s Modulus

$G$  
Shear Modulus

$g$  
Shear Parameter

$H$  
Thickness
LIST OF SYMBOLS

$I$  Moment of Inertia

$Im$  Imaginary Part of Complex Number

$i$  $\sqrt{-1}$

$K$  Kinetic Energy

$K_i$  Stiffness of $i^{th}$ Layer

$L$  Length

$m$  Mass

$p$  Wave Number

$Q$  Quality Factor

$Re$  Real Part of Complex Number

$t$  Time

$t_i$  Thickness of $i^{th}$ Layer

$U$  Strain Energy

$u$  Displacement in $x$-direction

$v$  Displacement in $y$-direction

$w$  Transverse Displacement

Superscripts

$*$  Complex Number
### Subscripts

- \( cl \)  Constraining Layer Property
- \( p \)  Plate Property
- \( s \)  System Property
- \( sl \)  Shear Layer Property
- \( 1 \)  Plate Property
- \( 2 \)  Shear Layer Property
- \( 3 \)  Constraining Layer Property
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Chapter 1

Introduction

1.1 Motivation

The most influential document in predicting the behavior of beams with constrained layer damping treatments was written in 1959 by Ross, Ungar, and Kerwin [1]. Without the assistance of large and efficient computers, it was necessary to develop a model that would predict both the damping and frequency of a vibrating beam with relatively simple calculations. Specifically, the paper discussed a theoretical modeling approach to the prediction of the damping and frequency of viscoelastically damped beams. This model is often referred to as the RUK model. The fundamental assumption of RUK theory is that the beam is of infinite length, allowing for the section of interest to have a purely sinusoidal mode shape. Such a beam behaves as if it has simply-supported boundary conditions. The assumptions and details of the RUK theory are discussed in Section 2.2.1. RUK theory has been an invaluable tool for structural designers concerned with mitigating stress or displacement amplitude in vibrating systems. For decades this model has been the standard for predicting properties of damped beams, and also for the characterization of viscoelastic material properties. Given observations of frequency, damping, and the mode shape of an experimental beam, it is possible to solve for the shear storage modulus and loss factor (measure
CHAPTER 1. INTRODUCTION

of damping) of the viscoelastic shear layer. However, for boundary conditions other than simply-supported, RUK theory can only approximate a solution. Designers of damping treatments are typically more interested in optimizing a damping treatment given design specifications such as geometry, density, and stiffness, than in predicting exact damping values.

With applications to disc brakes, computer hard drives, and turbine engines, the ability to accurately predict the damping and frequency of systems has become a necessity. Given the fact that damping treatments rarely resemble the simply-supported beam, it has become of interest to analyze treatments that have more complex boundary conditions and geometries. Specifically, structures that are more representative of a plate have gained in interest. While high speed computers now make it possible to use large, complex finite element models (FEM), it is still of interest to better understand the limitations of RUK theory as applied to geometries and boundary conditions that are more representative of current treatments. For simple models, finite element analysis (FEA) can sufficiently predict system damping and frequency when the material properties of the system are known. For the inverse problem of extracting unknown material properties for material characterization purposes, FEA alone is not sufficient. It would be very time consuming and difficult to extract material properties for material characterization purposes using FEA by itself. To do so would require an extensive iterative process to match both frequency and damping of experimental specimens and would produce multiple solutions. As will be shown in this thesis, both FEA and RUK analyses possess different, but significant, roles in future modeling of damped structures.

The extraction of damping in harmonic motion (i.e. modal loss factors) from finite element results can be accomplished by allowing the dissipative elements to be represented by a complex modulus. Extracting the complex frequency or frequency response will then allow for loss factor determination. The method of Modal Strain Energy (MSE) [2] has been introduced as a means of reducing the computational effort by eliminating the complex
numbers. The application of MSE to constrained layer damping treatments and sandwich beams presents a particular challenge, as materials with high loss factors are normally used. Moreover, while such materials normally have a low inherent modulus, their application in thin layers creates a high effective stiffness. This phenomenon has a significant influence on the distribution of energy within the structure, thereby impacting the accuracy of the method of MSE. It has been observed that when the dissipative element(s) has a high material loss factor (of order unity), the method of MSE over-predicts the modal loss factors [3, 4]. The source of such discrepancies will be examined, and a means for their reduction will be considered. Predicted loss factors found through a modification to MSE will be compared to those obtained from the complex valued response as found by finite element methods, to RUK solutions, and also to an exact solution for system parameters characteristic of symmetric sandwich cantilever beams optimized for maximum damping.

1.2 Literature Review

Methods for developing prediction and measurement capabilities of damped systems have been examined since the 1950s. Ross, Ungar, and Kerwin [1] proposed a methodology in 1959 that for decades has been well accepted in the damping world. The RUK model is formulated in terms of a complex bending stiffness of a three layer beam from which the frequency and damping can be determined. The RUK model assumes purely sinusoidal mode shapes and is most appropriate for simply-supported or infinitely long beams. Based on the underlying theory, it is evident that the RUK model is not applicable to plate-like structures with generic boundary conditions.

In the first effort to extend the concepts of RUK to other boundary conditions, DiTaranto [5] showed that a beam with a constrained layer damping treatment led to a sixth-order differential equation. Mead and Markus [6] derived such a sixth-order equation of motion in terms of transverse displacements of a beam with a single constrained layer damping
treatment and developed boundary conditions for several end conditions. However, exact solutions to boundary conditions other than simply-supported were not obtained. Like the RUK model, this derivation is based on Euler-Bernoulli beam theory and uses assumptions similar to that of the RUK model. Mead and Markus [7] provide some exact solutions for built-in, or clamped, ends. Rao [8] developed a sixth-order model by applying Hamilton’s Principle to system energies. As a result, numerical results were presented for sandwich beams with clamped-clamped, free-free, simply-supported, clamped-pinned, and cantilever boundary conditions. Rao’s differential equation, although derived by different means, is exactly the same as that of Mead and Markus [6]. Rao also developed algebraic formulas for the approximation of damping and frequency of damped beams.

Mead [9] developed a sixth-order differential equation of motion for the transverse displacements of a plate with a constrained layer damping treatment. The results for a simply-supported plate are very similar to his previous work with Markus [6], except that the influence of aspect ratio and a Poisson’s ratio is incorporated by use of the term \((1 - \nu^2)\). An eighth-order partial differential equation to describe the vibratory motion of sandwich plates was developed by He and Ma [10]. Noting that exact solutions were not possible for general boundary conditions, results were presented only for the simply-supported plate. An approach was described using an asymptotic solution of the governing equations to solve for boundary conditions other than simply supported. It is noteworthy to mention that results from the first order approximation of the asymptotic He and Ma [10] solution are the same as those from the method of MSE. Cupial and Niziol [11] derived a set of nine equations by using the principle of virtual work to develop the differential equation of motion for a free vibration system. The results derived from the equation of motion were then compared to a complex eigenvalue (CE) solution from a finite element model of a simply supported damped plate.

While an understanding of the mechanical principles remains important, today’s engineers are more apt to use FEM to predict the response of damped systems. Therefore, the
emphasis in research on constrained layer damping treatments has shifted to the application of finite element solutions to the three-layer damped system. Johnson and Kienholtz [2] formulated the method of Modal Strain Energy (MSE) and the CE solution to evaluate damped sandwich beams. The CE solution was compared to the MSE solution for the same configuration. Results were shown for both bending and torsional modes. Johnson and Kienholtz [2] also took the opportunity to evaluate the effectiveness of a finite element MSE solution for a simply-supported plate and compared the results to a sixth-order analytical expression derived by Abdulhadi [12]. The MSE results were in excellent agreement with the sixth-order formula for systems where the damping material loss factors were relatively small. Rongong [3] improved the method of MSE by using the magnitude of the complex shear modulus in the finite element computation of strain opposed to the real part only, and then computed the stored energy from the strains and the storage modulus. This modification is a significant improvement to the method of MSE. Torvik and Runyon [4] obtained loss factors by the method of MSE using ANSYS® finite element software. Results were then compared for the same structure, mesh density, and element type, to CE solutions obtained by Nastran, FE results from Soni and Bogner [13], predictions using Rongong’s modified MSE method [3], results derived from the sixth-order theory solution [14], and to other modifications to MSE. It is evident [4] that the method of MSE and the CE solutions are not necessarily in agreement.

While it can be seen that predicting the frequency and damping of a viscoelastically damped beam has been well investigated, only limited results on the influence of aspect ratio or boundary condition appear in literature. It is of interest to this research to evaluate the damping of plates with constrained layer damping treatments for a multitude of mode shapes, boundary conditions, and aspect ratios for both sandwich and thin constraining layer applications.
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1.3 Approach

The objective of this research is to investigate the effectiveness of RUK, MSE, and CE solutions in determining the frequency and damping of beams and plates for various mode shapes, aspect ratios, and boundary conditions of a constrained layer damping system for two constraining layer thicknesses. The geometries of interest are plates of various aspect ratios, with free edges and cantilever, free-free, simply-supported, and clamped-clamped end conditions. Results for aspect ratios between 0.5 and 14 will be investigated.

1.3.1 Ross, Ungar, Kerwin

Since the RUK model is the standard we are using for modeling simply-supported three-layer damped beams, it is important to compare the predictions from the RUK solution to those from alternative models. Depending on the boundary conditions and the aspect ratio, a partial restraint of the Poisson contraction can lead to an increase in effective stiffness. The maximum possible value of this effect would occur for a complete suppression of one component of the in-plane displacement (plane strain), in which case the stiffness would be increased by a factor of \((1 - \nu^2)\). Results obtained with a modification of the original RUK beam model incorporating this additional stiffness term, identified as the RUK plate model, will be computed and compared with results from the RUK beam model. Additionally, by using the effective length method proposed by Abdulhadi [12], both the RUK beam and adjusted plate model will be applied to symmetric boundary conditions other than simply-supported. It is also necessary to investigate the effectiveness of RUK modeling over a broad range of boundary conditions, mode shapes, and geometric parameters.

1.3.2 Method of Modal Strain Energy

It is evident in recent research that the method of MSE has limitations. It has been shown [3, 4] that if the material loss factor is near unity then the method of MSE can dramati-
cally over-predict system damping. To this end, research has been driven to modify MSE in order to more accurately predict the system response. Due to advantages in user interface, preprocessing, post-processing capabilities, and general availability, ANSYS® will be used to verify mode shape and solve for the strain energies necessary to calculate the MSE solutions provided in this research. Therefore, this research will include an in-depth comparison of MSE, modified MSE (MSE-Rongong) [3], CE solutions, and RUK models.

1.3.3 Complex Eigenvalue Solution

The CE solution has been proven [2,4] to demonstrate excellent agreement with sixth-order theory. Since finite element software does not utilize such simplifying assumptions of the RUK model as the neglect of shear in the face sheets, longitudinal stiffness of the shear layer, and rotary inertia; the well converged CE solution will be taken as the exact, and thus, the baseline solution. The justification for this will be demonstrated in Chapter 3. Since Nastran allows the user to enter a material loss factor as a direct material property input, and ANSYS® does not, Nastran will be used to develop the CE solutions presented in this work.

1.4 Overview and Contributions

1.4.1 Overview

It is the intent of this research to give insight to the limitations and applicability of current methods for predicting damping and frequency in three-layer viscoelastically damped beams and plates. The effect of boundary conditions and anti-clastic curvature on all models will be investigated by running test cases that include as many as six different aspect ratios for various boundary conditions. For baseline validation purposes, the first data set will include the configuration modeled by Johnson and Kienholtz [2], and verified by
CHAPTER 1. INTRODUCTION

(a) Cantilever  
(b) Free-Free  
(c) Simply-Supported  
(d) Clamped-Clamped

Figure 1.1: 3rd Bending Mode Shapes for Several Boundary Conditions

Torvik and Runyon [4].

The transverse displacement plots in Figure 1.1, show the difference in the 3rd bending mode shape due to the boundary conditions for a sandwich plate with an aspect ratio of 2. Although bending modes are the primary focus, the 2-stripe mode of a cantilever plate, Figure 1.2, will also be tracked for aspect ratios less than or equal to two. The bending modes have small variations in transverse displacements across the width (chord-wise direction) of the plate. However, the chord-wise variation in transverse displacement of 2-Stripe mode is significant and similar to the mode shape with free-free boundary conditions vibrating in its first natural mode. Although not an exact representation, modeling this plate mode shape as a free-free beam of length equal to the width of the chord would allow designers to...
better estimate the response using initial design specifications when investigating possible damping treatments. The extent to which this can be done successfully will be investigated.

Traditional cantilever boundary conditions on constrained layer damped beams or plates require the constraining layer to also be cantilevered. It may be possible to induce additional shear strain in the dissipative viscoelastic core by allowing the constraining layer to be free on all edges. The extent to which this changes system damping and frequency will also be characterized. A major objective of this research is to evaluate the limitations of RUK theory as applied to both beams and plates given a variety of boundary conditions.
1.4.2 Contributions of This Thesis

- This thesis will provide insight into the effectiveness of various models in the prediction of damping and frequency in damped systems.

- A collection of a broad range of data for the effects of boundary conditions and aspect ratio on loss factors and frequencies of damped beams and plates will be provided.

- A better understanding of the limitations of the RUK methodology will enable more accurate extraction of material properties from test specimens.

- Finite element results will demonstrate that it is possible to accurately model damping for the purpose of design.

- A MATLAB® program to generate a three-dimensional mesh for Nastran input files of beams and plates will be provided.
Chapter 2

Background

2.1 Undamped Systems

2.1.1 Beam Theory

The Euler-Bernoulli beam theory is commonly used for predicting the vibratory response of beams. Within its limitations, the Euler-Bernoulli theory predicts beam response very well. Since an understanding of beam theory and boundary conditions is important to this research, a review of Euler-Bernoulli beam theory is provided.

The details of assumptions behind the Euler-Bernoulli beam as stated by Inman [15], are listed below. The beam is considered to be:

- Uniform along its span, or length, and slender
- Composed of a linear, homogeneous, isotropic elastic material without axial loads
- Such that plane sections remain plane
- Such that the plane of symmetry of the beam is also the plane of vibration so that rotation and translation are decoupled
- Such that rotary inertia and shear deformation can be neglected
The Euler-Bernoulli beam equation of motion can be written as:

$$f(x,t) = \rho A(x) \frac{\partial^2 w(x,t)}{\partial t^2} + \frac{\partial^2}{\partial x^2} \left[ EI(x) \frac{\partial^2 w(x,t)}{\partial x^2} \right]$$  \hspace{1cm} (2.1)$$

The boundary conditions require that each end of the beam have specified moment \(\left( \frac{\partial^2 w(x,t)}{\partial x^2} \right)\) or slope \(\left( \frac{\partial w(x,t)}{\partial x} \right)\), and shear force \(\left( \frac{\partial^3 w(x,t)}{\partial x^3} \right)\) or displacement \(w(x,t)\). The boundary conditions that are evaluated in this research are the clamped-free (cantilever), free-free, clamped-clamped, and pinned-pinned (simply-supported). The cantilever boundary condition requires that \(w(x,t) = (\frac{\partial w}{\partial x}) = 0\) at the location \(x = 0\) and \((\frac{\partial^2 w(x,t)}{\partial x^2}) = (\frac{\partial^3 w(x,t)}{\partial x^3}) = 0\) at \(x = L\). The clamped-clamped boundary condition requires that \(w(x,t) = (\frac{\partial w}{\partial x}) = 0\) at \(x = 0\) and \(x = L\). Free-Free boundary conditions require \((\frac{\partial^2 w(x,t)}{\partial x^2}) = (\frac{\partial^3 w(x,t)}{\partial x^3}) = 0\) at \(x = 0\) and \(x = L\). The boundaries are said to be simply-supported if \(w(x,t) = (\frac{\partial^2 w(x,t)}{\partial x^2}) = 0\) at both \(x = 0\) and \(x = L\). The frequencies for the undamped Euler-Bernoulli beam can be identified through Equation 2.2, where \(\beta_n\) can be found from the eigenvalues for each of the boundary conditions. Table 2.1 is a list of the corresponding eigenvalues for each bending mode number and boundary conditions considered in this research. The eigenvalue of 0.0 for the first free-free mode has been omitted as it represents only a rigid body mode of zero frequency.

$$\omega_n = \beta_n^2 \sqrt{\frac{EI}{\rho A}}$$  \hspace{1cm} (2.2)$$

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<td>17.2787597</td>
<td>5(\pi)</td>
<td>17.2787597</td>
</tr>
</tbody>
</table>

Table 2.1: Euler-Bernoulli Eigenvalues
2.1.2 Plate Theory

Classical plate theory is based on a fourth-order differential equation of motion. However, it is not possible to analytically solve for the eigenfunctions for plates unless at least two opposite edges are simply-supported. The governing differential equation for a plate of uniform thickness as given by Equation 2.3.

\[
\rho H \frac{\partial^2 w(x,y,t)}{\partial t^2} + D \left[ \frac{\partial^4 w(x,y,t)}{\partial x^4} + 2 \frac{\partial^4 w(x,y,t)}{\partial x^2 y^2} + \frac{\partial^4 w(x,y,t)}{\partial y^4} \right] = q(x,y,t),
\]

(2.3)

Where:

\[
D = \frac{EH^3}{12(1 - \nu^2)}
\]

(2.4)

More often than not, numerical methods are used to solve for the response of a plate. Leissa [16] provides a plethora of solutions, in terms of dimensionless frequencies, for a wide variety of boundary conditions and aspect ratios. The boundary condition requirements are similar to those for the Euler-Bernoulli beam. Since it was not necessary to solve for any boundary conditions for this research, only the governing equation is shown. All plate modeling will be conducted using finite elements. For more details on classical plate theory see Leissa [16].

2.2 Damped Systems

There are two well understood methods of dissipating energy from vibrating structures using materials of high inherent damping, such as viscoelastic materials. These two methods are generally referred to as free layer and constrained layer damping treatments. In a free layer system the dissipative material is placed on the outside surface of the structure to be
damped. As the structure with the free layer treatment vibrates, the dissipative material undergoes mainly extensional, or axial, deformation. As the dissipative material is strained, deformations of molecular chains in the material may occur, which are thought to create internal heating [17] and thus a reduction in mechanical (vibratory) energy. In a constrained layer damping system (CLDS), the dissipative material behaves as the shear layer of the structure, sandwiched between the structure to be damped and a constraining layer (usually metal). It appears that Kerwin [18] was the first to present an analysis of the constrained layer damping system. He concluded that when placed between the plate to be damped and a constraining layer, viscoelastic materials dissipated energy by the shear motion induced into the viscoelastic material. It has been shown [1] that on an equal weight comparison the CLDS is more effective at reducing vibratory response than a free layer treatment.

System damping is typically reported as either a loss factor, $\eta_s$, or the quality factor, $Q = \frac{1}{\eta_s}$. The system loss factor is approximately equal to twice the critical damping ratio of a vibrating system at resonance and is proportional to the ratio of energy dissipated to energy stored, per cycle of vibration. In this work, material damping properties will only be reported in terms of loss factor, $\eta$, while system damping will be described as $\eta_s$. For the purpose of this research, analyses only of the CLDS will be considered and the dissipative material will be assumed to be a viscoelastic polymer, a dissipative material having both viscous and elastic properties that are dependent on both temperature and frequency.

2.2.1 Ross, Ungar, Kerwin

Ross, Ungar, and Kerwin [1] developed in 1959 a model generally referred to as the RUK model or theory. In this model, the displacements of the mid-planes of the three layer constrained layer damping system were calculated by setting the sum of the forces equal to zero and computing an effective bending stiffness for the section. Modeling the shear modulus as a complex number, a complex flexural rigidity term was then developed for the case of a beam undergoing forced transverse vibration, having infinite length and sinusoidal
Figure 2.1: RUK Nomenclature

mode shapes. In consequence, the model is most applicable to either simply-supported beams, or beams with other boundary conditions for mode numbers high enough that a significant part of the structure is undergoing purely sinusoidal displacements so that the mode shape, or transverse displacement can be approximated as $A \sin(px)$. Damping is incorporated into the model by use of the complex modulus approach. The intent here is to show the simplicity of the RUK model. Therefore, a detailed derivation of the algebraic expressions will not be given. For insight into the derivations see Reference [1]. Figure 2.1 provides standard nomenclature of a three-layer CLDS. The following equations are for the RUK beam model. Complex valued quantities are denoted by an asterisk.

$$D^* = \frac{K_2^* \left(H_{21} - \frac{H_{31}}{2}\right) + (K_2^* H_{21} + K_3^* H_{31}) g^*}{K_1 + \frac{K_2^*}{2} + g^* \left(K_1 + K_2^* + K_3^*\right)}$$

(2.5)

where:

$$K_i = E_i H_i$$

(2.6)

$$K_2^* = H_2 E_2^*$$

(2.7)

$$E_2^* = E_2 (1 + i\eta)$$

(2.8)

Similarly:
\[ G_2^* = G_2 (1 + i\eta) \quad (2.9) \]
\[ g^* = \frac{G_2^*}{K_3H_2p^2} \quad (2.10) \]

Here, \( p \) represents the wave number, and \( G_2 \) is the real part of the shear modulus of the shear layer. Furthermore, \( \eta \) is the loss factor of the viscoelastic material. Therefore, \( G_2^* \) is a complex number representing the complex shear modulus of the viscoelastic material.

The mass per unit area of the system is:
\[ m = \rho_1H_1 + \rho_2H_2 + \rho_3H_3 \quad (2.12) \]

Because \( K_2^*, g^* \), and \( D^* \) are all complex numbers, the flexural rigidity \( B^* \), is also complex. From this complex stiffness, the system frequency can easily be obtained by Equation 2.13.
\[ \omega_n = \sqrt{\frac{p^4 \text{Re}(B)}{m}} \quad (2.13) \]

Also, the system loss factor, \( \eta_s \), can be found from Equation 2.14.
\[ \eta_s = \frac{Im(B)}{Re(B)} \]  

(2.14)

\[ p = \frac{n\pi}{L} \]  

(2.15)

\[ L_{EFF} = \left( \frac{\pi}{\beta_n L} \right) L \]  

(2.16)

\( L_{EFF} \) is the effective length of a simply-supported beam with the same frequency and system loss factor in its first bending mode and is substituted for \( L/n \) in Equation 2.15 for the cases when the RUK approximation for boundary conditions other than simply-supported is desired. Here \( \beta_n L \) is the eigenvalue derived from Euler-Bernoulli beam theory as described in Section 2.1.1, of the desired mode and boundary condition as tabulated in Table 2.1.

The RUK equation adjusted for maximum stiffness enhancement (RUK plate model) is identical with the exception that:

\[ K_i = \frac{E_i}{\left(1 - \nu_i^2\right)}H_i \]  

(2.17)

Notice that the adjusted RUK plate model is independent of the aspect ratio. This will not allow for a sensitivity assessment of the adjusted RUK model to aspect ratio, but may provide a better estimate of loss factor and frequency than the RUK beam model for structures that are more plate-like (that is, with low aspect ratios). The simplicity, and thus the value, of the RUK model is now evident.

### 2.2.2 Higher Order Plate Theory

The significant difference between classical beam theory and higher order plate theory for constrained layer damping systems is that the longitudinal strain in the cover fibers (the
constraining layer) is reduced because of deformations in the shear layer. As in the case of the constrained layer damping treatments on beams, it is the relative axial deformation of the plate with respect to the constraining layer that induces the large shear strains into the shear layer. Thus, axial displacements become significant to the overall system response and are the additional degrees of freedom that raise the order of the system. With the addition of a shear and constraining layer to beams and plates, the neutral bending axis moves from the geometric center of the plate towards, or even into, the shear layer.

2.2.3 Finite Element Models

For design purposes, engineers are more concerned with the ability to model in finite element software packages than to understand sixth-order differential equations. For this reason, finite element software packages such as ANSYS® 8.1 and Nastran 2004 will be used to model damped systems, providing feedback both to the designer and the researcher on the effectiveness of each model and its corresponding parameters.

2.2.3.1 Method of Modal Strain Energy

The method of modal strain energy (MSE) enables a significant simplification in the prediction of damping with finite elements. Software such as ANSYS® can be used to calculate, tabulate, and add the energies in individual elements. In this method, the viscoelastic material is modeled with only a real modulus and the imaginary part is neglected. The results of MSE analysis provided in this research were obtained by using three-dimensional solid structural elements in ANSYS®. Specifically, eight noded (SOLID45) bricks were used, each element having three translational degrees of freedom per node.

For the method of MSE, the system loss factor (system damping) is directly proportional to the ratio of the energy dissipated in the viscoelastic elements to the energy stored in the entire system through one cycle of vibration. This ratio is then multiplied by the loss factor of the viscoelastic material. The system loss factor is shown in Equation 2.18.
Here the elemental strain energies for the $i^{th}$ layers are $U_i$. Both the strain energies in the dissipative viscoelastic layers and the non-dissipative components are considered.

$$\eta_s = \frac{\sum \eta_i U_{i,\text{Dissipative}}}{\sum U_{i,\text{Dissipative}} + \sum U_{i,\text{Non-Dissipative}}} \quad (2.18)$$

Much debate has gone on as to which value of the viscoelastic modulus to use. Traditional MSE only uses the real part of the shear modulus. The Rongong [3] modification to the method of MSE adjusts the shear modulus. In his modification, the magnitude of the shear modulus is used to solve for the strain energies in the system. This is accomplished by defining a parameter $\alpha = \sqrt{1 + \eta^2}$ which is multiplied by the real part of the shear modulus and the product used as the shear modulus in the finite element model. For material loss factors near unity, Equation 2.9 shows that the magnitude of the shear modulus may be significantly greater than the real part alone. After solving for the strain energies, the stored energies in the dissipative materials are divided by $\alpha$, essentially correcting the tabulated strain energies. These corrected energies are then used in Equation 2.18 to calculate the system loss factor.

### 2.2.3.2 Complex Eigenvalue Solution

When damping is added to the stiffness matrix by using a complex modulus, the complex eigenvalue solver extracts complex roots. These roots are representative of the complex system frequency, and system damping can then be extracted from the complex system frequency by Equation 2.19.

$$\eta_s = \frac{\text{Im} (\omega^2)}{\text{Re} (\omega^2)} \quad (2.19)$$

The complex eigenvalue solutions were accomplished by using MSC Nastran version 2004.0.0 [19] from MSC Software Corporation. While ANSYS® also provides complex eigenvalue analysis, it does not allow the user to input a complex valued material prop-
erty, as does Nastran. The disadvantage of Nastran is the user interface. Nastran does not have its own preprocessor, so it is necessary to create the geometry, nodes, and element connectivity in other software. For this purpose, a finite element mesh generator was designed in MATLAB®. This code, applicable to three-layer beam and plate geometry, is provided in Appendix A. The eight noded CHEXA solid element was used for the Nastran complex eigenvalue solutions. The Nastran solver used a complex Lanczos method, which is comprised of three steps. First the matrix is reduced to tridiagonalized form. Next, the tridiagonal problem is solved. Finally, the eigenvectors are computed. The details of this eigenvalue method can be seen in [19].

2.3 Test Cases

Viscoelastic material properties are temperature and frequency dependent. Typically, the properties of most interest to the damping community are the real storage modulus and loss factor. Normally, these properties are plotted as a function of temperature and frequency on a nomogram. See Figure 2.2 for a sample nomogram of a 3M Corporation 467 pressure sensitive adhesive [20]. Since material properties are arbitrary in this study, the storage modulus and loss factor of the shear layer are held constant. The Young’s modulus, shear modulus, and loss factor for all test cases are assumed to be 300psi, 100psi, and 1.0, respectively. This is a hypothetical material, as a real viscoelastic material would not have the same properties at all temperatures and frequencies. Since the objective of this study is to compare predictions by different methods for a range of modes, aspect ratios, and boundary conditions, rather than to provide design information for a specific configuration, it is of no additional benefit to incorporate temperature and frequency dependence into the models. This approach is consistent with the analysis of Johnson and Kienholtz [2].
2.3.1 Sandwich Model

The most efficient of all three layer CLDSs is the symmetric sandwich configuration. Shear strain is maximized in the viscoelastic core when the constraining layer is of the same stiffness as the base, and the system neutral axis is centered through the middle of the shear layer. Thus, for a given substrate and shear layer, the system loss factor is maximized when the constraining layer is of the same material and geometry as the substrate, so that the viscoelastic layer is sandwiched between two identical face sheets. In this document, the term sandwich is reserved for only those three layer systems in which the constraining and base layers are of the same geometry and material properties. Often, in the design of damped systems, it is of utmost importance to maximize the damping effectiveness for a given material. For this reason, the symmetric sandwich system will be thoroughly evaluated. For all sandwich data, the base and constraining layers are taken to be aluminum of 0.060” thick. The elastic modulus is defined as 10E6 psi and the weight density is 0.1 lbs/in$^3$. The shear storage modulus and loss factor of the 0.005” viscoelastic core are 100 psi and 1.0 respectively. The weight density and Poisson’s ratio are also taken to be 0.035 lb/in$^3$ and 0.5 respectively. Viscoelastic materials are assumed to be nearly incompressible,
corresponding to a Poisson’s ratio near 0.5, and a Young’s storage modulus three times larger than the shear modulus. The shear layer geometry and material properties are the same for all test cases in this research. Both beam and plate-like structures will be evaluated as a function of mode shape, boundary condition, and aspect ratio. In all cases the plate was taken to be 7” in length. Table 2.2 provides summary of the configuration and material properties.

### Table 2.2: Configuration Table for all Test Cases

<table>
<thead>
<tr>
<th>Configuration</th>
<th>Base Layer Properties</th>
<th>Constraining Layer Properties</th>
<th>Shear Layer Properties</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$E$ (psi)</td>
<td>$\rho$ (lb/in^3)</td>
<td>Thickness (in)</td>
</tr>
<tr>
<td>Sandwich</td>
<td>1.00E+07</td>
<td>0.1</td>
<td>0.06</td>
</tr>
<tr>
<td>Thin CL</td>
<td>1.00E+07</td>
<td>0.1</td>
<td>0.06</td>
</tr>
</tbody>
</table>

2.3.2 Thin Constraining Layer Model

While the sandwich configuration may enable the optimal use of a given shear layer, it is not always desirable to double the weight and thickness of the system to be damped. In systems that are weight sensitive, such as aerospace or automotive structures, the optimal design may be more driven by weight or even cost. For such situations, a thin constraining layer may often be the best solution. It is therefore desirable to compare the various methods for predicting system damping for such configurations.

The base and constraining layer are taken to be aluminum 0.060” and 0.005” thick, respectively. The elastic modulus of both is defined as 10E6 psi and the weight density 0.1 lbs/in^3. This will produce a base to constraining layer bending stiffness ratio of 1728. The shear storage modulus and loss factor of the 0.005” viscoelastic core are 100 psi and 1.0 respectively. The weight density and Poisson’s ratio are again taken to be 0.035 lb/in^3 and 0.5 respectively, so that the Young’s storage modulus is three times that of the shear modulus, or 300psi. The viscoelastic geometry and material properties are the same for
all test cases. Both beam and plate-like structures will be evaluated as a function of mode shape, boundary condition and aspect ratio. In all cases the plate was taken to be 7” in length. Table 2.2 provides summary of the configuration and material properties.

### 2.3.3 Boundary Conditions

It is not always possible to accurately model real world applications. In attempting to do so, difficulties often arise in the ability to properly match boundary conditions. However, it is practical to study the behavior and limitations of basic boundary conditions and then apply these lessons learned to real world applications. It is this fundamental understanding of design parameters and effects of boundary conditions that motivate this research. Without a complete understanding of design parameters and modeling limitations, damping design would be an imprecise process.

In keeping with traditional beam theory analysis, the conditions are enforced for opposing ends, while the other edges are assumed to be free. These boundary conditions are conducive to mode shapes that resemble beam bending modes. For this reason, this research is focused on the first several bending modes of all test cases, although the 2-Stripe mode of the cantilevered plate is also considered. The boundary conditions chosen for the bare material in this study are cantilever, clamped-clamped, simply-supported, and free-free for opposing ends and free on the other two sides. Typically, the constraining is assumed to be free of axial load on all edges except in the case of a clamped end. For both clamped-free (cantilever) and clamped-clamped test cases, the constraining layer has the same boundary condition as the base material. A special test case was created to allow the constraining layer to be free at the fixed end of a cantilever. It was desired to see if the free constraining layer would allow for the induction of biaxial stress into the shear layer, and if the effect would be greater for the more plate-like configurations. This in turn would increase the system damping.
2.3.4 Investigation of Aspect Ratio

It is commonly accepted that the bending stiffness of a plate is about 10% larger than a beam of the same length, thickness, and material properties. This can be demonstrated through Equation 2.17. For predicting the behavior of beams and plates, however, this relationship is somewhat binary. Either the geometry is a beam whose bending stiffness is represented in Equation 2.1 or it is a plate whose bending stiffness is represented in Equation 2.3. What is not clearly understood is the aspect ratio where a plate starts behaving as a beam and vice versa. Since beam theory is simpler and more easily applied, it is of interest to establish the conditions for which a beam is an adequate model of a plate.

2.3.5 The 2-Stripe Mode

As mentioned previously in Section 1.4.1, this research is focused on the first five bending modes of vibration, with the exception being the 2-stripe mode of a plate. The 2-stripe mode is a cantilever mode shape that resembles the first free-free beam mode across the chord. See Figure 1.2 for a picture of this mode shape. This mode shape has been of great interest to design and test engineers in the aerospace industry. For test purposes, the cantilever boundary condition is the most easily replicated experimentally. However, a common problem with the cantilever boundary condition is that the location of maximum strain (thus the area of crack initiation) is at the clamp interface. Applying instrumentation to measure the strain at or near a clamp is very difficult. Visually detecting cracks at or within the clamp interface is also difficult. For fatigue testing, it is important to evaluate the strain displacement relationship on the test article during the test, and to determine when a crack has initiated. In the 2-Stripe mode shape the maximum strain in the mode is at the tip of the free end. Strain gages are easily mounted near the tip, and cracks can be more easily detected. Because of these features, the 2-Stripe mode has been of great interest in fatigue testing and characterization.

Airfoil design engineers are also interested in this mode shape for another reason. On
a low aspect ratio airfoil, the 2-Stripe can be an easily excited mode shape. Although the points of maximum displacement and strain are on the blade tips, tip displacements are normally much less than those in the lower cantilever modes of high aspect ratio. This reduces the effectiveness of traditional tip dampers and creates a critical need for other damping mechanisms. For these reasons, the 2-Stripe mode is evaluated for the cantilever boundary conditions where the aspect ratio is less than or equal to 2. The frequency and mode number for aspect ratios greater than 2 become so large they will not be considered at this time.
Chapter 3

Results and Discussion

3.1 Validation of Finite Element Results

In order to validate the results of this research and ensure accurate finite element results, mesh convergence studies were conducted. The first study evaluated the number of elements in both the span-wise and chord-wise directions. A 7” square sandwich plate geometry was chosen as the test case. All the elements used in this research were three-dimensional, eight noded, solid brick elements, with each node having three translational degrees of freedom. The material properties and geometry of this configuration can be found in Section 2.3.1. The first model consisted of 10 elements in both the span-wise (x) and chord-wise (z) directions, while each layer contained only one element through the thickness. The number of elements was doubled in both the span-wise and chord-wise directions for each of the two successive iterations. The complex eigenvalue results of the first five bending modes are presented in Table 3.1. The percent difference was calculated by subtracting the 40x40 mesh results from the 20x20 mesh results and then dividing by the 40x40 results. It can be seen that the difference is at most 2.49% for frequency and 1.85% for loss factor and was greatest for the 4th bending mode. In all cases, increasing the number of degrees of freedom reduced the frequency as expected. Results obtained
with the 40x40 mesh were accepted as being sufficiently well converged.

The next study involved the number of elements through the thickness of each layer. For this study a 42x3 mesh for a cantilevered sandwich beam of aspect ratio 14 was used. Again, the frequencies and loss factors for the first five bending modes were evaluated. These results were also computed from the Nastran complex eigenvalue solution. It can be observed from Table 3.2 that the difference between results obtained with one or three elements through the thickness is small. The percent differences were calculated from the mesh results of the models with both two elements and three elements through the thickness of each layer. The results suggested that the use of one element through the thickness of each layer (40x40x1 mesh) was sufficient. The difference between the two elements per layer and one element per layer appeared to be about 90% of the difference between three elements and one element per layer. The increase in frequency with increasing elements through the thickness, although very small, is unexpected and may be attributed to the

<table>
<thead>
<tr>
<th>1 Element Through Thickness</th>
<th>2 Element Through Thickness</th>
<th>3 Element Through Thickness</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency (Hz)</td>
<td>Loss Factor</td>
<td>Frequency (Hz)</td>
</tr>
<tr>
<td>68.08</td>
<td>0.2004</td>
<td>68.18</td>
</tr>
<tr>
<td>306.26</td>
<td>0.2143</td>
<td>306.90</td>
</tr>
<tr>
<td>757.02</td>
<td>0.1481</td>
<td>758.75</td>
</tr>
<tr>
<td>1416.00</td>
<td>0.0865</td>
<td>1419.66</td>
</tr>
<tr>
<td>2306.95</td>
<td>0.0555</td>
<td>2313.69</td>
</tr>
</tbody>
</table>

Table 3.2: Convergence Study of Elements Through Thickness
excessive element aspect ratios with thinner elements. Therefore, it was decided to model only one element through the thickness of each layer to reduce computation time and file size.

Although it was determined that 40 elements through a length of 7” would be sufficient, 42 elements were chosen as the number of element divisions for a 7” length. The resulting element edge length of 1/6” in the span and chord-wise directions enabled the same mesh density (elements/unit length) in both directions as well as aspect ratios represented by simple integers and rational fractions. The element aspect ratios for a given configuration are not changed as the geometry aspect ratio changes. Therefore, the square plate models have the same number of elements in both the length and width of the plate. For future reference, the mesh density for this research will be presented in the form AxBxC where A, B, and C represent the number of elements through the length, width, and thickness of each layer respectively. The aspect ratio will be defined as the length in the x-direction divided by the width in the z-direction.

Since the CE solutions will be considered the exact solution for each model configuration, it is necessary to provide some validation of these results. The CE solution will be compared to the RUK simply-supported solution since we know RUK provides an exact solution for the case of a simply-supported beam. The CE model for a 7” sandwich beam with an aspect ratio of 14 was solved in Nastran and compared to the solution from the RUK beam model. The FE beam was modeled with an element mesh of 42x3x1. The results can be seen in Table 3.3. The results correlated very well for the case of the simply-supported beam. The largest difference was found to be 2.27% for the loss factor of the 5th bending mode. Some difference is to be expected as the aspect ratio (14) for the CE results is less than that of a beam (∞). The CE solution satisfactorily agrees with the RUK beam prediction for the simply-supported beam.

Since the ANSYS® MSE results will be compared to Nastran CE solutions, it is necessary to compare the accuracy of the two different eigensolvers. Frequencies obtained from
**Simply-Supported - Sandwich Beam**

<table>
<thead>
<tr>
<th>Frequency (Hz)</th>
<th>Loss Factor</th>
<th>Frequency (Hz)</th>
<th>Loss Factor</th>
<th>Frequency (Hz)</th>
<th>Loss Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>154.29</td>
<td>0.3052</td>
<td>156.16</td>
<td>0.2983</td>
<td>1.20</td>
<td>2.33</td>
</tr>
<tr>
<td>491.64</td>
<td>0.1918</td>
<td>495.40</td>
<td>0.1895</td>
<td>0.76</td>
<td>1.23</td>
</tr>
<tr>
<td>1035.75</td>
<td>0.1066</td>
<td>1044.78</td>
<td>0.1054</td>
<td>0.86</td>
<td>1.14</td>
</tr>
<tr>
<td>1794.77</td>
<td>0.0651</td>
<td>1819.48</td>
<td>0.0641</td>
<td>1.36</td>
<td>1.63</td>
</tr>
<tr>
<td>2769.90</td>
<td>0.0434</td>
<td>2828.06</td>
<td>0.0424</td>
<td>2.06</td>
<td>2.27</td>
</tr>
</tbody>
</table>

Table 3.3: Comparison of Simply-Supported Results

A Nastran solution with a loss factor equal to zero were compared with frequencies obtained by MSE using ANSYS®. A 7” cantilever beam with an aspect ratio of 14 (42x3x1 mesh) was evaluated in this investigation. The geometry and material properties (except for the loss factor) were the same as previously discussed in Section 2.3.2. It can be seen in Table 3.4 that for, an undamped system, both sets of frequencies are in excellent agreement.

**Cantilever Beam - AR = 14**

<table>
<thead>
<tr>
<th>Frequency (Hz)</th>
<th>MSE</th>
<th>CE</th>
<th>% Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>42.42</td>
<td></td>
<td>42.41</td>
<td>0.03</td>
</tr>
<tr>
<td>251.54</td>
<td></td>
<td>251.45</td>
<td>0.04</td>
</tr>
<tr>
<td>683.91</td>
<td></td>
<td>683.59</td>
<td>0.05</td>
</tr>
<tr>
<td>1321.40</td>
<td></td>
<td>1320.47</td>
<td>0.07</td>
</tr>
<tr>
<td>2177.20</td>
<td></td>
<td>2174.95</td>
<td>0.10</td>
</tr>
</tbody>
</table>

Table 3.4: Finite Element Eigensolver Comparison

Frequencies and loss factors for the first three cantilever modes of a sandwich beam of the geometries and materials described in Section 2.3.1 have been obtained by Sun, Sankar, and Rao [14] and by Soni and Bogner [13] using finite element methods. Additionally, Sun [14], used Rao’s 6th-order theory [8] to obtain frequencies and loss factors for the
### Chapter 3. Results and Discussion

#### Table I: Modal Loss Factors, Constrained Layer on Cantilever Beam

<table>
<thead>
<tr>
<th>Mode</th>
<th>LF (Sun)</th>
<th>LF (Soni)</th>
<th>LF 6th Ord</th>
<th>Freq (Sun)</th>
<th>Freq (Soni)</th>
<th>Freq 6th Ord</th>
</tr>
</thead>
<tbody>
<tr>
<td>First</td>
<td>0.1997</td>
<td>0.2022</td>
<td>0.2019</td>
<td>67.5</td>
<td>67.4</td>
<td>67.4</td>
</tr>
<tr>
<td>Second</td>
<td>0.2117</td>
<td>0.2177</td>
<td>0.2180</td>
<td>298.1</td>
<td>302.8</td>
<td>317.0</td>
</tr>
<tr>
<td>Third</td>
<td>0.1452</td>
<td>0.1502</td>
<td>0.1500</td>
<td>752.8</td>
<td>748.6</td>
<td>762.0</td>
</tr>
</tbody>
</table>

Table 3.5: Results from Previous Analysis of Sandwich Cantilever Beam

The same configuration. These findings are given in Table 3.5. These values may be compared with values obtained in this study. These are the first three lines of Table 3.6. The agreement is seen to be satisfactory.

#### 3.2 Loss Factors and Frequencies

Results will be presented as plots of loss factor versus frequency, normalized frequency versus aspect ratio, and loss factor versus aspect ratio. Data was obtained only for the discrete values of modal frequency. Six plots are presented for each boundary condition. The first three plots are MSE, RUK beam, RUK plate, and CE solutions presented as loss factor versus frequency for aspect ratios of 14, 3.5, and 1. Plot four (second row, right) is a comparison of the MSE, MSE-Rongong, and CE finite element solutions for the case of the square plate (AR=1). Plots five and six illustrate the dependence of normalized frequency and loss factor on aspect ratio. The frequency in plot five has been normalized to the frequency of the system with a geometric aspect ratio of 14. The data points have been joined by smooth curves to more clearly illustrate the trends. Values in-between modal frequencies should not be taken as representative of loss factors at other than modal frequencies.
3.2.1 Sandwich Results

The results of all evaluated boundary conditions and aspect ratios for the sandwich configuration are presented in Tables 3.6-3.11. Results for selected aspect ratios are given in Figures 3.2-3.6.

The CE solution of the first five bending modes of a cantilever beam for aspect ratios of 0.5, 1, 2, 3.5, 7, and 14, are presented in Figure 3.1. From these plots, it is apparent that the effect of changing the width from 0.5” to 7” for a specimen 7” in length had a relatively small effect on frequency and system loss factor. All six plots are essentially superpositions of each other.

For FE modeling purposes, it was necessary to properly model the boundary conditions for all test cases discussed in this research. The FE constraints for the cantilever models were applied by setting all displacements equal to zero at the root of the base, shear, and constraining layer areas. It was not necessary to apply constraints to the free end in order to model a free end condition. This is consistent with the zero displacement, zero
CHAPTER 3. RESULTS AND DISCUSSION

slopes assumption of the clamped boundary condition and zero moments, zero shear force assumption of the free end.

The first three plots (left to right, top to bottom) in Figure 3.2 illustrate the MSE and CE finite element solutions, as well as results from the RUK beam and plate models for a 7” cantilever specimen with aspect ratios 14, 3.5, and 1. The fourth plot is a comparison of the three (MSE, CE, MSE-Rongong) finite element models for the square (AR=1) plate sandwich configuration. The MSE results for bending modes 1 and 2 do not sufficiently predict the system loss factor. The MSE and both RUK models over-predict damping in the first two bending modes, with the largest discrepancy occurring at mode 1. For the given configuration, the optimum (largest) system loss factor should occur between modes 1 and 2 as indicated by the peak in the damping curve. The MSE solution completely misses the peak damping configuration, while the other three models capture the presence of a peak at nearly the same frequency (mode number) value. The difference in frequency for all models appears to be about the same percentage for all modes. The frequency and loss factor results for bending modes 3-5 are in reasonable agreement for all models. The MSE and CE solutions appear nearly identical for modes 3-5. This is consistent with previous work [4] indicating that MSE tends to over-predict damping near the damping peak, when the shear layer material loss factor is near unity. It is also notable that although the RUK beam and plate models over-predict damping for the first two modes, they both under-predict damping in modes 3-5. The effect of aspect ratio appears to be negligible for the cantilever boundary condition. The fourth plot shows the significant improvement of the MSE-Rongong model over the traditional MSE model for predicting damping for all modes. The MSE-Rongong appears to slightly under-predict damping for the lower modes and to be in excellent agreement with CE results for modes 3-5, as expected [4]. Plot five shows a slight frequency dependence on aspect ratio. The frequency increases as the aspect ratio decreases. Plot six shows only a slight change in loss factor for aspect ratios 0.5 through 3.5. The loss factor changes very little for aspect ratios 3.5 through 14. Note
Figure 3.2: Frequency and Loss Factor Plots for Sandwich Configuration Cantilever Boundary Condition
that plots five and six show some irregularities near aspect ratios of 0.5 and 1. The reason for this is not understood at this time. The finite element tabular data for this configuration can be seen in Table 3.6. Here, CF represents the cantilever boundary condition; el and ew represent the number of elements through the length and width of the model. Results given for the 2-Stripe mode will be discussed in a later section.

Although a traditional cantilever CLDS constrains the constraining layer, it is of interest to measure the effect on frequency and damping when the constraining layer is left free. The cantilever with free constraining layer CLDT has a slightly different boundary condition than the cantilever model previously discussed. For this model, only the base layer displacements were set to zero at the root end area. For this configuration, no finite element constraints were applied to the shear or constraining layers.

The results for the cantilever sandwich configuration in which the constraining and shear layers were left free can be seen in Figure 3.3. Notice the significant improvement of the MSE model for modes 1 and 2. Here, the MSE model was able to locate the peak damping configuration between modes 1 and 2. Again, like the traditional sandwich cantilever models previously discussed, the MSE model over-predicts damping for the first two modes. Both RUK models over-predict damping for modes 1-2, and under-predict for modes 3-5, just as with the traditional cantilever sandwich previously mentioned. Since the RUK model does not allow the user to free up the constraining layer, the RUK predictions for these results are the same as the previous plots in Figure 3.2. Although properly predicting the peak damping to be near mode 2, the exact location of the peak is errantly shifted to the left for both RUK models. The fourth plot indicates that MSE over-predicts and MSE-Rongong slightly under-predicts damping for the first two modes, but both are in excellent agreement with the CE results for modes 3-5, as previously seen. As in the case with the traditional cantilever CLDT, no model appears to be significantly sensitive to aspect ratio. Plots five and six have the same trends as the traditional cantilever boundary condition: a slight frequency increase as aspect ratio decreases and a loss factor that remains virtually
### Sandwich - Cantilever Boundary Condition

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Table 3.6: Tabulated Frequency and Loss Factor Results for Sandwich Configuration Cantilever Boundary Condition
Figure 3.3: Frequency and Loss Factor Plots for Sandwich Configuration Cantilever Boundary Condition with Free Constraining Layer
### Sandwich - Cantilever w/ Free Constraining Layer

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Table 3.7: Tabular Data of Loss Factor and Frequency for Sandwich Configuration Cantilever Boundary Condition with Free Constraining Layer
constant for aspect ratios of 3.5 through 14. The tabulated FE results for the cantilever with free constraining layer (CF-FCL) boundary condition are listed in Table 3.7. Results given for the 2-Stripe mode will be discussed in a later section.

The plots in Figure 3.4 illustrate the results for the free-free boundary condition. There were no constraints on any of the layers in the model, allowing for accurate representation of the zero moment and zero shear force requirements of the free boundary condition.

Notice that for the configuration used in these computations no damping peak is found for this boundary condition. Unlike the previous results, the RUK models consistently under-predict damping for all of the first five bending modes. It appears that the RUK may converge to the exact (CE) solution as the mode number increases. The MSE solution over-predicts damping for mode 1 by 30%-40%, but agrees well for modes 2-5. The distance between node lines for mode 1 free-free is larger than that of mode 2 of a cantilever beam of the same free length. There appears to be no sensitivity with the models to the change in aspect ratio for the free-free boundary condition. The fourth plot indicates that the MSE-Rongong is in excellent agreement for all modes. Plot five in Figure 3.4 illustrates the slight dependence of frequency on aspect ratio. Again, some unexplained irregularities are seen for aspect ratios less than 2. The loss factor appears to be constant for aspect ratios of 3.5 or greater. The tabulated FE data for the free-free (FF) end conditions can be found in Table 3.8.

In order to model the simply-supported boundary condition in the finite elements, it was necessary to constrain the bottom plate corner of both ends to a zero transverse (y) displacement. This is consistent with the zero transverse displacements, zero bending moment requirements for the simply-supported (pinned-pinned) end condition. It is important not to constrain the span-wise displacements. Constraining either the x or z components effectively would render the model too stiff to be representative of the simply-supported boundary conditions, and an over-prediction in frequency would be evident. The simply-supported results for the sandwich configuration can be seen in Figure 3.5.
Figure 3.4: Frequency and Loss Factor Plots for Sandwich Configuration Free-Free Boundary Condition
Table 3.8: Tabulated Frequency and Loss Factor Results for Sandwich Configuration Free-Free Boundary Condition

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Figure 3.5: Frequency and Loss Factor Plots for Sandwich Configuration Simply-Supported Boundary Condition
The results plotted in Figure 3.5 show that, as with the free-free models, the simply-supported configuration does not capture a peak in the damping versus frequency plots, indicating that the optimum configuration for these parameters is not achieved for any mode. Like the free-free configuration, the simply-supported mode 1 results have a larger system loss factor than any of the cantilever modes. The most significant difference in results were evident in the mode 1 MSE solution. As expected, both RUK models are in excellent agreement for all modes, however, it is not apparent which RUK model is a better approximation for any of the aspect ratios. The MSE-Rongong results are a marked improvement over traditional MSE for the mode 1 results. Again, the results for this boundary condition did not appear to be sensitive to aspect ratio for any of the models. Again, the normalized frequency increases as aspect ratio increases. The frequencies of bending modes 3-5 appear to be about 4.5% larger than the beam (AR=14) frequencies when the aspect ratio is 0.5. The loss factor does not demonstrate a significant dependence on aspect ratio. Note that the curves of plots five and six are much smoother than the previous plots near the low aspect ratio region. The FE tabular data for the simply-supported (SS) boundary condition can be seen in Table 3.9.

The clamped end conditions for the clamped-clamped models were described as previously mentioned for the clamped-free case, except that both ends are now clamped. It was necessary to do so in order to achieve the zero displacement, and zero slope requirements for each end of the clamped-clamped boundary condition. The clamped-clamped results for the sandwich configuration can be seen in Figure 3.6. The mode 1 loss factors are significantly lower than for any other boundary condition. This is attributed to the lack of shear strain at the clamped ends of the shear layer. Also, this configuration and boundary condition are further from the peak damping configuration. The free-free and simply-supported boundary conditions allow for large shear strains in the shear layer at the ends of the structure. Without these large shear strains, the clamped ends will significantly reduce the amplitude of the mode 1 system loss factors. Both RUK models over-predict
### Sandwich - Simply Supported

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Table 3.9: Tabulated Frequency and Loss Factor Results for Sandwich Configuration

Simply-Supported Boundary Condition
Figure 3.6: Frequency and Loss Factor Plots for Sandwich Configuration Clamped-Clamped Boundary Condition
damping for all modes in this test case. It does appear that the RUK loss factor results converge to the CE loss factor as the mode number increases, however, the RUK frequency predictions do not converge. They appear to be in error by about the same percentage for all modes. The RUK loss factor predictions for the first two modes appear also appear to be significantly different than the CE solutions. The MSE results, however, are in excellent agreement with CE for all modes and aspect ratios. All the FE models appear to be in excellent agreement for the clamped-clamped boundary condition and are insensitive to aspect ratio. Plot five indicates that all modes demonstrate similar dependence of normalized frequency on aspect ratio. The curves in plot six are very flat, indicating little to no dependence on aspect ratio. The tabular data for the clamped-clamped (CC) FE results can be seen in Table 3.10.

Because all loss factor versus frequency plots for a given model have the same shape for every aspect ratio, it does not appear that any boundary condition for the sandwich test cases is significantly sensitive to aspect ratio. However, the frequencies for all boundary conditions did increase slightly as aspect ratio decreased. Also, the dependence of loss factor on aspect ratio did not appear to be significant. It appears that the RUK model with enhanced stiffness (RUK plate model) does not provide a significant improvement over the RUK beam model. For all aspect ratios, the RUK models are most effective for the simply-supported boundary condition and reasonably effective for mode numbers greater than 2 for the cantilever, and less satisfactory for all other boundary conditions. Agreement at the higher modes is better as mode shapes begin to resemble pure sinusoids for all boundary conditions. For the given sandwich configuration, the MSE finite element model appears to be adequate for modes 3-5, with reasonable effectiveness for mode 2, for all boundary conditions and aspect ratios. The Rongong modification to the method of MSE is equally effective for all boundary conditions and mode shapes for the square plate geometry. Since aspect ratio has a very small effect on all evaluated boundary conditions for the sandwich configuration, it can be expected that the Rongong modification would produce a reason-
**Sandwich - Clamped Clamped**

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Table 3.10: Tabulated Frequency and Loss Factor Results for Sandwich Configuration Clamped-Clamped Boundary Condition
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Table 3.11: Tabulated Frequency and Loss Factor Results of a Square Plate Sandwich Configuration all Evaluated Boundary Conditions

ably accurate model for all aspect ratios. Results obtained using Rongong’s modification to the method of MSE are compared with results obtained with traditional MSE and CE solutions for an aspect ratio of 1 are shown in Table 3.11.
3.2.2 Thin Constraining Layer Results

The thin constraining layer plots were generated from models having the same boundary conditions, aspect ratios, shear layer properties, plate properties, and constraining layer material as the previously discussed sandwich models. The constraining layer thickness was the only difference between the sandwich and thin constraining layer (TCL) models. Since it was determined that aspect ratio was not a significant factor in the results of the sandwich configuration, TCL data is given for aspect ratios of 14, 3.5, and 1, for all models. Both the cantilever models (CF and CF-FCL) also have results for the 7, 2, and 0.5 aspect ratio in the tabulated data. The results for all evaluated boundary conditions for the configuration with a thin constraining layer are presented in Tables 3.12-3.17. Results for selected aspect ratios are given in Figures 3.7-3.11.

The cantilever plots for the TCL configuration are significantly different than the same plots for the sandwich configuration. The results indicate a peak in the damping curve for all plots of the cantilever boundary condition and can be seen in Figure 3.7. Although damping is over-predicted at the peak amplitude, the RUK models correctly show the trends of the CE solution. There is no significant difference between the frequency and loss factor predictions of the two RUK models. The MSE prediction appears to over-predict damping in the first three bending modes (around the peak), which is consistent with previous findings. Even the MSE-Rongong square plate frequency and loss factor predictions show greater error than the cantilever sandwich results. Much like the cantilever sandwich configuration, the normalized frequency curves are smooth for aspect ratios 3.5 through 14. However, the curves for the thin constraining layer configuration have more significant irregularities for low aspect ratios. The loss factor has only a slight dependence on aspect ratio with no distinguishable pattern. The tabulated data for the thin constraining layer cantilever (CF) models can be seen in Table 3.12.

The results for the TCL cantilever with free constraining layer configuration as found in Figure 3.8 are significantly different than the same results for the sandwich configura-
Figure 3.7: Frequency and Loss Factor Plots for Thin Constraining Layer Configuration
Cantilever Boundary Condition
### Thin Constraining Layer - Cantilever

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Table 3.12: Tabulated Frequency and Loss Factor Results for Thin Constraining Layer Configuration Cantilever Boundary Condition
Figure 3.8: Frequency and Loss Factor Plots for Thin Constraining Layer Configuration
Cantilever Boundary Condition with Free Constraining Layer
tion. The CE, and therefore the exact solution, appears to be sensitive to the system aspect ratio. The shape of the CE plots change, indicating a significant dependence on aspect ratio for modes 1 and 2. Consistent with previous results, the MSE model over-predicts the system loss factor near the area of peak damping. The RUK models under-predict damping for mode 1 and over-predict for modes 2 and 3. Both the MSE and RUK solutions appear to accurately capture damping for modes 4-5. The CE solution suggests two peaks for the square plate, where the MSE solution suggests the two peaks for all aspect ratios. Although the MSE-Rongong model is consistently better than the traditional MSE for the square plate configuration, the traditional MSE method appears to be more accurate for modes 4-5. Although the loss factor amplitudes consistently decrease with decreasing aspect ratio, it appears that the mode 2 loss factor decreases significantly for the square plate TCL configuration. Perhaps the most significant observation is the gross difference in peak damping location between the RUK models and the CE solution. Trying to optimize a configuration based on RUK predictions would lead to a less than optimal configuration. The normalized frequency curves for the case of the free constraining layer are very different for each mode. However, all modes have at least one peak in the normalized frequency plot. Except for the peak of mode 1, the loss factor does not appear to be sensitive to aspect ratio. The tabulated FE data for this configuration (CF-FCL) can be seen in Table 3.13.

For the free-free TCL test case, the models again predict the damping peak location to the left of the actual location. See Figure 3.9 for free-free TCL plots. Other than mode 1, the RUK solutions under-predict damping for all aspect ratios. No model with the TCL on a free-free beam appears to be sensitive to aspect ratio. MSE over-predicts loss factors near the peak damping configuration, but agrees reasonably well for modes 3-5. The MSE-Rongong model appears to under-predict damping as much as MSE over-predicts for the square plate configuration. Modes 1, 3, 4, and 5, indicate a significant dependence on aspect ratio for both the normalized frequency and loss factor. Also, the percent change in frequency and loss factor is much larger for this particular model than any previous
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Table 3.13: Tabulated Frequency and Loss Factor Results for Thin Constraining Layer Configuration Cantilever Boundary Condition with Free Constraining Layer
Figure 3.9: Frequency and Loss Factor Plots for Thin Constraining Layer Configuration Free-Free Boundary Condition
### Thin Constraining Layer - Free Free

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Table 3.14: Tabulated Frequency and Loss Factor Results for Thin Constraining Layer Configuration Free-Free Boundary Condition

As expected, the RUK results for the simply-supported CLDT are in excellent agreement for all modes, and are each independent of aspect ratio as seen in Figure 3.10. The MSE results indicate an over-prediction in system damping near the peak damping location. Both the MSE and MSE-Rongong models are reasonably accurate for modes 3-5 and the MSE-Rongong also agrees well with mode 1. Aspect ratio did not appear to influence the results of any approximate methods significantly. The fifth plot demonstrates a slight frequency dependence on aspect ratio. The loss factor does not appear to be dependent on aspect ratio. The tabulated results for the simply-supported (SS) CLDT can be seen in Table 3.15.

The plots of loss factor versus frequency for the clamped-clamped CLDT with a TCL appears to be noticeably different than the clamped-clamped sandwich configuration. The results can be seen in Figure 3.11. The RUK models are in excellent agreement for the mode
Figure 3.10: Frequency and Loss Factor Plots for Thin Constraining Layer Configuration
Simply-Supported Boundary Condition
Figure 3.11: Frequency and Loss Factor Plots for Thin Constraining Layer Configuration
Clamped-Clamped Boundary Condition
CHAPTER 3. RESULTS AND DISCUSSION

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Table 3.15: Tabulated Frequency and Loss Factor Results for Thin Constraining Layer Configuration Simply-Supported Boundary Condition

1 solution, but over-predict damping for all other modes. MSE slightly over-predicts damping for the first mode, but is in excellent agreement for modes 2-5, much like the clamped-clamped sandwich configuration. It appears that although the MSE-Rongong agrees very well for the square plate configuration, the traditional MSE model may be more representative of the exact solution. Again, there appears to be no sensitivity to aspect ratio for the approximate methods. Again, the frequency increases slightly as aspect ratio decreases, but the loss factor does not have a significant dependence on aspect ratio. The tabulated frequency and loss factor results for the clamped-clamped (CC) TCL configuration can be seen in Table 3.16.

The TCL data indicates that there are larger discrepancies in the prediction of system loss factor than for the comparable sandwich configuration. Even for the clamped-clamped and free-free boundary conditions, where the sandwich MSE results were in very good agreement with the CE solution, the MSE results for the TCL configuration had a noticeably larger over-prediction in system loss factor.

Overall, the frequency increased as aspect ratio decreased for all boundary conditions.
with the thin constraining layer configuration. Also, the free-free boundary condition had the largest frequency changes with aspect ratio. Near the region of low aspect ratio (0.5 through 2) the thin constraining layer configuration had much more prominent peaks than the sandwich configuration for the normalized frequency plots. The presence of such irregularities, typically found near the square plate geometry, was confirmed by additional calculations for the cantilever plates with thin constraining layers and aspect ratios of 5/6 and 7/6. The loss factor of this configuration showed significant dependence on aspect ratio only for the free-free boundary condition. This behavior in the low aspect ratio region is not well understood. Some really interesting and very different results appear for the case of the TCL cantilever with a free constraining layer (TCL CF-FCL). This is the only case that has significant dependence on aspect ratio, and the models appear to predict two damping peaks when loss factor is plotted against frequency. It is also noteworthy that the MSE-Rongong solutions agree much better when the CLDT is a sandwich opposed to having a thin (significantly less stiff) constraining layer. As previously mentioned, the sandwich configuration is more effective than the same system with a thin constraining

Table 3.16: Tabulated Frequency and Loss Factor Results for Thin Constraining Layer Configuration Clamped-Clamped Boundary Condition

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### Table 3.17: Tabulated Frequency and Loss Factor Results of a Square Plate Thin Constraining Layer Configuration all Evaluated Boundary Conditions

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This can be seen by observing the difference in system loss factor between the sandwich and TCL configuration, given the same boundary condition and aspect ratio. The MSE-Rongong, MSE, and CE frequency and loss factor results for an aspect ratio of 1 can be seen in Table 3.17.
3.2.3 Ross, Ungar, Kerwin Results

As expected the RUK models are the most effective for predicting the loss factor and frequency of a simply-supported beam. The RUK models were also equally effective with either constraining layer configuration. The modification to the RUK model did not appear to affect the loss factor prediction capability significantly. Perhaps the slight improvement in frequency prediction would make the modified RUK more reliable for material property characterization when the experimental specimen aspect ratio is less than or equal to 3.5. For the models with an aspect ratio greater than or equal to 3.5, the RUK beam model is the superior model for predicting frequency results. The CF-FCL configuration dramatically decreases the ability of the RUK models to predict not only the magnitude of system loss factor, but also the location. It would be difficult to optimize a CLDT if the peak damping were errantly located by any predictive model. Caution must be used when using effective lengths to model boundary conditions other than simply-supported with an RUK model. The complete set of RUK frequency and loss factor results can be seen in Tables 3.18 and 3.19.

3.2.4 Discussion of the 2-Stripe Mode

The results for the 2-Stripe mode may be seen in Figures 3.12 and 3.13. The 2-Stripe mode shape is shown in Figure 1.2. There are no noticeable differences in the results when comparing the free constraining layer configuration to the traditional cantilever configuration for either the sandwich or thin constraining layer. The Rongong modification to MSE provided a very good estimate of frequencies and loss factors in the case of the square plate.

The RUK modeling of the 2-Stripe mode consists of a free-free beam of length equal to the plate width, in its first bending mode. Both RUK models significantly under-predict the 2-Stripe frequency for the sandwich and thin constraining layer configuration. RUK loss factor predictions were satisfactory only for the square sandwich configuration. While
### Table 3.18: Tabulated RUK Frequency and Loss Factor Results for Sandwich Configuration Cantilever Boundary Condition

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### RUK Thin Constraining Layer Data

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Table 3.19: Tabulated RUK Frequency and Loss Factor Results for Thin Constraining Layer Configuration Cantilever Boundary Condition
### RUK 2-Stripe Predictions

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<td>Sandwich</td>
<td>2S</td>
<td>1043.08</td>
<td>0.1059</td>
<td>1098.15</td>
<td>0.0963</td>
<td>2</td>
</tr>
<tr>
<td>Sandwich</td>
<td>2S</td>
<td>300.45</td>
<td>0.2572</td>
<td>312.75</td>
<td>0.2448</td>
<td>1</td>
</tr>
<tr>
<td>Sandwich</td>
<td>2S</td>
<td>98.28</td>
<td>0.2814</td>
<td>101.73</td>
<td>0.2898</td>
<td>0.5</td>
</tr>
<tr>
<td>Thin CL</td>
<td>2S</td>
<td>974.64</td>
<td>0.0499</td>
<td>1028.76</td>
<td>0.0470</td>
<td>2</td>
</tr>
<tr>
<td>Thin CL</td>
<td>2S</td>
<td>258.07</td>
<td>0.0601</td>
<td>272.04</td>
<td>0.0617</td>
<td>1</td>
</tr>
<tr>
<td>Thin CL</td>
<td>2S</td>
<td>67.21</td>
<td>0.0278</td>
<td>71.06</td>
<td>0.0304</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Table 3.20: Tabulated RUK Frequency and Loss Factor Solutions for Free-Free Modeling of 2-Stripe Mode

It may be possible to get a rough approximation of damping and frequency by modeling the plate 2-Stripe plate mode as a free-free beam in the first bending mode, an accurate approximation is not likely. The results were dependent on aspect ratio, and there did not seem to be a distinguishable pattern. The tabulated FE data for the 2-Stripe mode was listed in Tables 3.2, 3.7, 3.12, and 3.13 as the 2S mode. The raw RUK data can be seen in Table 3.20.
CHAPTER 3. RESULTS AND DISCUSSION

Figure 3.12: Comparison Plots of Cantilever 2-Stripe Mode for Sandwich Configuration
Figure 3.13: Comparison Plots of Cantilever 2-Stripe Mode for Thin Constraining Layer Configuration
Chapter 4

Conclusion

4.1 Summary

A study of boundary condition and aspect ratio effects of constrained layer damping treatments (CLDT) applied to beams and plates has been provided. The effects of aspect ratio on the frequency and damping predictions of Ross, Ungar, and Kerwin (RUK) original and modified models, finite element methods using complex eigenvalues, modal strain energy (MSE), and Rongong’s modification to MSE, were included. Both sandwich and thin constraining layer configurations were investigated for the first five bending modes of vibration and for the 2-Stripe mode. Additionally, the influence of relaxing the constraining layer at the root of cantilever beams and plates was considered. The complex eigenvalue solutions were used as the baseline to which results from the more approximate methods were compared.

Overall, the system frequencies increased as the aspect ratio decreased for all bending modes, boundary conditions, and constraining layer configurations. This indicated that the bending stiffness of these plates with CLDT is effectively higher than that of beams with similar damping treatments. For several boundary conditions, significant irregularities in the dependence of frequency on aspect ratio were observed for geometrical aspect ratios
less than 2. In general, loss factors were found to diminish slightly as aspect ratio was reduced. The most notable exception was for the case of the free-free thin constraining layer configuration.

The RUK results were obtained by replacing the factor of $\pi$ in the shear parameter of the RUK solution by the eigenvalue corresponding to the desired mode and boundary condition. The RUK models were most effective for the simply-supported boundary condition, as expected. For cantilever modes 3-5, the RUK model agrees well with the complex eigenvalue system loss factor for both the thin and sandwich constraining layer configurations. For the free-free and clamped-clamped boundary conditions, the RUK model provides a reasonable approximation of system damping only for modes 4 and 5. An accurate prediction of system frequency is necessary for an accurate extraction of material properties from experimental data. Because of the relatively low modulus of the shear layer, a small frequency discrepancy in system response can be misinterpreted as a large change of the shear modulus for the shear layer. Generally, the RUK model with stiffness enhancement (RUK plate model) predicted more accurately the frequency of plates with an aspect ratio equal to or less than 3.5. In consequence, the RUK beam model should be used for test specimens with aspect ratios greater than 3.5.

Although the MSE frequency results slightly under-predict frequency, MSE frequency results were in excellent agreement for all aspect ratios, boundary conditions, and mode numbers. The method of MSE provided accurate loss factor predictions for modes not near the peak damping for all traditional boundary conditions for both constraining layer thicknesses. MSE over-predicts damping in all cases, but only significantly near a damping peak. For the sandwich configuration, such a peak occurs only for the cantilever boundary condition, whereas for the thin constraining layer, a peak was found for all boundary conditions except clamped-clamped.

The Rongong modification to MSE was evaluated for all boundary conditions and modes for both the sandwich and thin constraining layers for the case of the aspect ratio
equal to 1. The Rongong modification to the method of MSE appeared to be a significant improvement for all mode shapes and boundary conditions for the sandwich configuration. Because the MSE-Rongong uses the magnitude of the shear storage modulus, it over-predicted frequency for all modes and boundary conditions. In all cases, the MSE-Rongong model under-predicted system damping. For the sandwich configuration, this difference is small and the Rongong modification to MSE provides a significant improvement over traditional MSE using only the real part of the shear storage modulus. For the thin constraining layer, however, the Rongong modification leads to an under-prediction of loss factors near the damping peak which are approximately equal to the over-predictions of traditional MSE. At the higher mode numbers the errors, although negligible, are greater than those of MSE.

There were no noticeable differences in the ability of the approximate methods to predict frequency or damping of the 2-Stripe mode when comparing the free constraining layer configuration to the traditional cantilever configuration for either the sandwich or thin constraining layer. The Rongong modification to MSE provided a very good estimate of frequencies and loss factors in the case of the square plate. It does not appear that modeling the 2-Stripe plate mode as a RUK free-free beam yields consistently satisfactory results. Both the RUK beam and plate models significantly under-predict the 2-Stripe frequency for the sandwich and thin constraining layer configuration. Loss factor predictions were satisfactory only for the square sandwich configuration. Although the loss factor predictions may provide a rough approximate of the loss factor, designing a damping treatment with these approximations would not be advised. The results were dependent on aspect ratio, and there did not seem to be a distinguishable pattern.

Except for the case of the first mode for the thin constraining layer configuration, the relaxation of the constraining layer led to a decrease in system damping. For the cantilever beam with a free, thin, constraining layer, a second peak appears in the damping versus frequency plot as aspect ratio approaches unity. When the constraining layer is much less
stiff than the base layer, no model agrees well with the complex eigenvalue solution for the first three (near the damping peak) bending modes. All models are in reasonable agreement with the complex eigenvalue solution for modes 4 and 5. Figures 3.3 and 3.8 indicate that both RUK models over-predict the system damping by as much as 50% near the largest damping peak. The RUK predicted damping peak locations are also errant regardless of aspect ratio for this configuration. This would significantly affect damping design for this configuration.

4.2 Recommendations

There appears to be a potential for significant improvement to applications involving RUK modeling when the intent is to extract material properties from empirical data. The results of this research indicate that RUK analysis can continue to be used for all modes of the simply-supported boundary conditions. RUK can also be used effectively for all other boundary conditions. However, it is recommended that the results from this thesis be applied to RUK modeling for more accurate results. That is to say, for cantilever testing, the 1st and 2nd bending modes of vibration should not be trusted for RUK analysis. Also, the RUK beam model should be applied to structures that have an aspect ratio greater than or equal to 3.5. The RUK model with enhanced stiffness may provide better estimates of frequency when the aspect ratio is less than or equal to 3.5.

It has also been demonstrated that finite elements can successfully be used to model damped systems. Nastran is expensive and lacks post and preprocessing capabilities. However, its complex eigenvalue solution served as the baseline solution in this work because of its agreement with 'exact' analytical solutions. In environments where a complex eigenvalue solution is not obtainable, the method of MSE and Rongong’s modification to MSE have a place in finite element modeling. This type of modeling can be achieved on less expensive, all inclusive, more user friendly software. It is recommended that Rongong’s
modification to modal strain energy be used for FEA when modeling sandwich-like structures. When constrained layer damping structures that are more representative of a thin constraining layer configuration, the method of MSE and MSE-Rongong can be used to bound the solution for design purposes.

Although promising results and trends were found, caution should be used when extrapolating this research to configurations not demonstrated. The results provided in this research are at this point applicable to the limited set of boundary conditions studied. This includes the side conditions which were assumed to be free for all plate configurations. The data generated in this work were for only two constraining layer stiffness configurations and might not be applicable to other configurations. The results of this research are also limited to structures with damping treatments having full coverage on only one side of the structure. It is not recommended to extrapolate these results to structures with partial coverage damping treatments, or treatments applied to multiple sides.

4.3 Future Work

The application of RUK modeling to other boundary conditions would be very significant. One might investigate modifications, or introduce co-efficients to the method of effective lengths in order to enable the successful application of RUK modeling to more general boundary conditions. Although not a trivial study, perhaps modifications to the shear and/or geometric parameter could be made to the RUK model, bringing the RUK damping and frequency solutions into agreement with the CE solutions.

A thorough investigation of the phenomenon that caused the irregularities in the plots of normalized frequency and loss factor versus aspect ratio for low aspect ratio (0.5 through 2) geometries should be considered. There may, perhaps, be a condition in which closely spaced modes influence the performance of these damped plates.

In order to more fully evaluate the effect of aspect ratio, it would be of interest to
research models and configurations for which boundary condition constraints were applied to more than 2 edges. While this research focused on bending and 2-Stripe modes, it would be of interest to evaluate the influence of boundary conditions and aspect ratio on torsional and stiff-wise modes as well.

For the purpose of this work, shear layer thickness and material properties were not important. However, a future study of other shear layer geometries and properties, would enable a better extrapolation of these findings to future modeling and testing.
Appendix A

MATLAB® Mesh Program

This program generates the .dat input files for NASTRAN®.

% meshy.m
2 % 26 August 2004
% meshy.m generates an 8-noded brick mesh for beams and plates
4 % Tommy George and Brian Runyon
6 clear
8 clear all
10 close all

% user defined lengths
12 % l=length in x-direction
14 % w=width in y-direction
16 % h1=thickness of layer 1 in z-direction
18 % h2=thickness of layer 2 in z-direction
20 % h3=thickness of layer 3 in z-direction
22
16 l=7;
AR=.5;
18 w=.5;
19 h1=.06;
20 h2=.005;
21 h3=.06;

% user defined number of elements per edge
24 % el=number of elements in x-direction
APPENDIX A. MATLAB® MESH PROGRAM

% ew=number of elements in y-direction
% eh1=number of elements across layer 1
% eh2=number of elements across layer 2
% eh3=number of elements across layer 3

el=42;
ew=3;

eh1=1;
ch2=1;
ch3=1;

% Setting boundary conditions
% Set bcond=1 for 4F
% Set bcond=2 for Cantilever
% Set bcond=3 for CCFF
% Set bcond=4 for 4C
% Set bcond=5 for SSFF
% Set bcond=6 for 4S
% Set bcond=7 for Cantilever with FFFF CL
bcond=5;

46 totalnodes=(ew+1)*(el+1)*(eh1+eh2+eh3+1);
% dofs=6*totalnodes;
48 M = repmat(int8(0), dofs, dofs);
% K= repmat(int8(0), dofs, dofs);
50 for i=1:totalnodes
    grid(i,1)=i;
52 end

54 totaels=(ew*el)*(eh1+eh2+eh3);

56 i=1;
z=0;
58 for zplane=1:eh1+eh2+eh3+1,
    x=0;
60 for xplane=1:el+1,
    y=0;
62 for yplane=1:ew+1,
    grid(i,2)=x;
    grid(i,3)=y;
    grid(i,4)=z;
68 end
66 end
APPENDIX A. MATLAB® MESH PROGRAM

y = y + w / ew;
i = i + 1;
end
x = x + l / el;
end
if zplane < eh1 + eh2 + eh3 + 1
ztemp = z + h3 / eh3;
end
if zplane < eh1 + eh2 + 1
ztemp = z + h2 / eh2;
end
if zplane < eh1 + 1
ztemp = z + h1 / eh1;
end
z = ztemp;
end
for i = 1: total els,
elements (i, 1) = i;
end
i = 1;
nodes layer = (ew + 1) * (el + 1);
mat layer = 1;
for z layer = 1: eh1 + eh2 + eh3,
if z layer > eh1
mat layer = 2;
end
if z layer > eh1 + eh2
mat layer = 3;
end
for x layer = 1: el,
for y layer = 1: ew,
elements (i, 2) = mat layer;
elements (i, 3) = (y layer + (x layer - 1) * (ew + 1) + nodes layer * (z layer - 1));
elements (i, 4) = (y layer + (el + 1) * (ew + 1) + nodes layer * (z layer - 1));
elements (i, 5) = (y layer + ew + 2 + (x layer - 1) * (ew + 1) + nodes layer * (z layer - 1));
elements (i, 6) = (y layer + ew + 1 + (x layer - 1) * (ew + 1) + nodes layer * (z layer - 1));
elements (i, 7) = (y layer + (x layer - 1) * (ew + 1) + nodes layer * (z layer - 1));
elements (i, 8) = (y layer + 1 + (x layer - 1) * (ew + 1) + nodes layer * (z layer));
elements (i, 9) = (y layer + ew + 2 + (x layer - 1) * (ew + 1) + nodes layer * (z layer));
elements (i, 10) = (y layer + ew + 1 + (x layer - 1) * (ew + 1) + nodes layer * (z layer));
APPENDIX A. MATLAB® MESH PROGRAM

\[ i = i + 1; \]

```
i = i + 1;
end
end
```

```
\% Creating xtra columns for NASTRAN

elem = [ elements(:,1:8) elements(:,1) elements(:,1) elements(:,9:10)];
\% results = [grid];
\%
\% Create output file
\%
fid = fopen(’filename.dat’,’w’);
fprintf(fid,’$FILE_NAME=d:\models\runyon\filename.dat’n’);
fprintf(fid,’$n’);
fprintf(fid,’TIME=200
’);
fprintf(fid,’SOL=107’n’);
fprintf(fid,’CEND’);
fprintf(fid,’TITLE=Brian’s Sandwich Model’n’);
fprintf(fid,’SUBTITLE=Complex Eigenvalue Solver’n’);
fprintf(fid,’ECHO=NONE’n’);
fprintf(fid,’CMETHOD=2’n’);
fprintf(fid,’$DISP(PRINT,PUNCH)=ALL’n’);
fprintf(fid,’$STRESS=ALL’n’);
fprintf(fid,’SUBCASE 1’n’);
fprintf(fid,’LABEL=UNTITLED’n’);
fprintf(fid,’SPC=1’n’);
fprintf(fid,’$LOAD=1’n’);
fprintf(fid,’BEGIN BULK’n’);
fprintf(fid,’EIGC,2,CLAN,MAX,40’n’);
fprintf(fid,’PARAM,K6ROT,1.0’n’);
fprintf(fid,’PARAM,AUTOSPC,YES’n’);
fprintf(fid,’PARAM,COUPMASS,1’n’);
fprintf(fid,’PARAM,TINY,0.0000001’n’);
fprintf(fid,’$2345678123456781234567812345678123456781234567812345678’n’);
\%
\% Grid numbers and coordinates
\%
```
```
fprintf(fid,’GRID %6.0f %06.5f %06.5f %06.5f %06.5f
’,grid’);
fprintf(fid,’GRID %6.0f %06.5f %06.5f %06.5f
’,grid’);
```
```
fprintf(fid,’$n’);
fprintf(fid,’$CONNECTIVITIES’n’);
```
% element numbers and connectivity
fprintf(fid,'CHEXA%6.0f%6.0f%6.0f%6.0f%6.0f%6.0f%6.0f%6.0f%6.0f%6.0f%6.0f%6.0f
');
fprintf(fid,'PSOLID
');
fprintf(fid,'PSOLID
');
fprintf(fid,'PSOLID
');
fprintf(fid,'$23456781234567812345678123456781234567812345678123456781234\n');

% Cantilever
if bcond==2
  cantilever=find(grid(:,2)==0);
cant=size(cantilever);
umcantnodes=[cant(1,1)];
  for i=1:1:numcantnodes
    ja=cantilever(i,1);
    fprintf(fid,'SPC%6.0f123456\n',ja);
i=i+1;
  end
end

% Clamped–Clamped Free–Free
if bcond==3
  clamp1=find(grid(:,2)==0);
  clamp2=find(grid(:,2)>1);
clamp3=size(clamp1);
umclampnodes=cant3(1,1);
for i = 1 : 1 : numclampnodes
    jb = clamp1(i,1);
    fprintf(fid, 'SPC
%6.0f
123456\n', jb);
    i = i + 1;
end
for i = 1 : 1 : numclampnodes
    jc = clamp2(i,1);
    fprintf(fid, 'SPC
%6.0f
123456\n', jc);
    i = i + 1;
end

if bcond == 4
    clamp1 = find(grid(:,2) == 0);
    clamp2 = find(grid(:,2) >= 1 - .001);
    clamp3 = size(clamp1);
    clamp5 = find(grid(:,3) == 0);
    clamp6 = find(grid(:,3) >= w - .001);
    clamp7 = size(clamp5);
    numclampnodes2 = clamp7(1,1)
    numclampnodes = clamp3(1,1)

% X-axis
for i = 1 : 1 : numclampnodes
    jd = clamp1(i,1);
    fprintf(fid, 'SPC
%6.0f
123456\n', jd);
end
for i = 1 : 1 : numclampnodes
    je = clamp2(i,1);
    fprintf(fid, 'SPC
%6.0f
123456\n', je);
end

% Y-axis
for i = 1 : 1 : numclampnodes2
    jf = clamp5(i,1);
    fprintf(fid, 'SPC
%6.0f
123456\n', jf);
end
for i = 1 : 1 : numclampnodes2
    jg = clamp6(i,1);
    fprintf(fid, 'SPC
%6.0f
123456\n', jg);
end
APPENDIX A. MATLAB® MESH PROGRAM

79

end

% 2S2F  Simply−Supported Free−Free

if bcond==5

pin1=find((grid(:,2)==0)&(grid(:,4)==0));
numpinnodes=max(size(pin1));

pin2=find((grid(:,2)>1-.001)&(grid(:,4)==0));
numpinnodes2=max(size(pin2));

for i=1:1:numpinnodes
   jh=pin1(i,1);
   fprintf(fid,'SPC %6.0f 3
',jh);
end

for i=1:1:numpinnodes2
   ji=pin2(i,1);
   fprintf(fid,'SPC %6.0f 3
',ji);
end

end

% 4S Simply−Supported Simply−Supported

if bcond==6

pin3=find((grid(:,2)==0)&(grid(:,4)==0));
numpinnodes3=max(size(pin3));

pin4=find((grid(:,2)>1-.001)&(grid(:,4)==0));
numpinnodes4=max(size(pin4));

pin5=find((grid(:,3)==0)&(grid(:,4)==0));
numpinnodes5=max(size(pin5));

pin6=find((grid(:,3)>w-.001)&(grid(:,4)==0));
numpinnodes6=max(size(pin6));

for i=1:1:numpinnodes3
   jj=pin3(i,1);
   fprintf(fid,'SPC %6.0f 3
',jj);
end

for i=1:1:numpinnodes4
   jk=pin4(i,1);
   fprintf(fid,'SPC %6.0f 3
',jk);
end

for i=1:1:numpinnodes5
   jl=pin5(i,1);
   fprintf(fid,'SPC %6.0f 3
',jl);
end

end
for i = 1:1:numpinodes
    jm = pin6(i,1);
    fprintf(fid, 'SPC1%6.0f123456\n', jm);
end

% Cantilever
if bcond == 7
    canty = find((grid(:,2)==0) & (grid(:,4) <= h1));
    canty2 = size(canty);
    numpantynodes = [cantly2(1,1)];
    for i = 1:1:numpantynodes
        jn = canty(i,1);
        fprintf(fid, 'SPC1%6.0f123456\n', jn);
        i = i + 1;
    end
end

fprintf(fid, 'ENDDATA\n');

fclose(fid);

ee = elements(:,1);
    ee = [ee, elements(:,3:10)];
plotmesh(grid, ee)
    axis equal
Bibliography


